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HAMILTONIAN COLORINGS

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Abstract:

This paper studies the Hamiltonian coloring and Hamiltonian chromatic number for different graphs. The main results are: 1. For any integer n greater than or equal to three, Hamiltonian chromatic number of C_n is equal to $n-2$. 2. G is a graph obtained by adding a pendant edge to Hamiltonian graph H , and then Hamiltonian chromatic number of G is equal to $n-1$. 3. For every connected graph G of order n greater than or equal to 2, Hamiltonian chromatic number of G is not more than one increment of square of $(n-2)$.

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Key words: Chromatic number, Hamiltonian coloring, Hamiltonian chromatic number, pendant edge, spanning connected graph.

1 Introduction:

Generally in a $(d-1)$ radio coloring of a connected graph G of diameter d , the colors assigned to adjacent vertices must differ by at least $d-1$, the colors assigned to two vertices whose distance is 2 must differ by at least $d-2$, and so on up to antipodal vertices, whose colors are permitted to be the same. For this reason, $(d-1)$ radio colorings are also referred to as antipodal colorings.

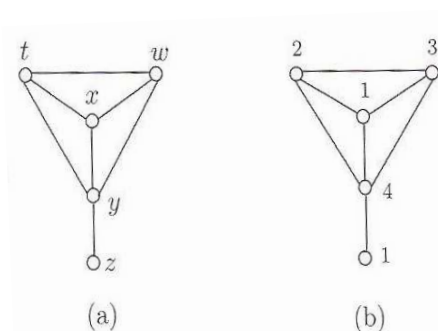
In the case of an antipodal coloring of the path P_n of order $n \geq 2$, only the two end-vertices are permitted to be colored the same. If u and v are distinct vertices of P_n and $d(u, v) = i$, then $|c(u) - c(v)| \geq n - 1 - i$. Since P_n is a tree, not only is i the length of a shortest $u-v$ path in P_n , it is the length of the only $u-v$ path in P_n . In particular, i is the length of a longest $u-v$ path

The detour distance $D(u, v)$ between two vertices u and v in a connected graph G is defined as the length of a longest $u-v$ path in G . Hence the length of a longest $u-v$ path in P_n is $D(u, v) = d(u, v)$. Therefore, in the case of the path P_n , an antipodal coloring of P_n can also be defined as a vertex coloring c that satisfies.

$D(u, v) + |c(u) - c(v)| \geq n - 1$, for every two distinct vertices u and v of P_n .

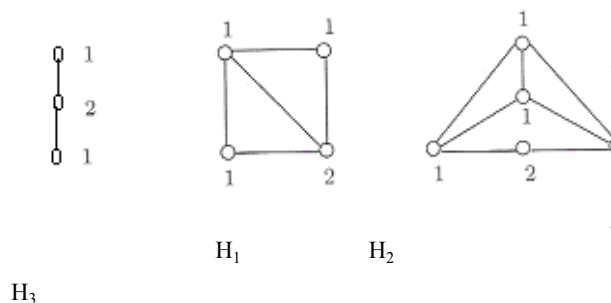
1.1 Definition: Vertex coloring c that satisfy were extended from paths of order n to arbitrary connected graphs of order n by Gary Chartrand, Ladislav Nebesky, and Ping Zhang. A **Hamiltonian coloring** of a connected graph G of order n is a vertex coloring c such that, $D(u, v) + |c(u) - c(v)| \geq n - 1$, for every two distinct vertices u and v of G . The largest color assigned to a vertex of G by c is called the **value** of c and is denoted by $hc(c)$. The **Hamiltonian chromatic number** $hc(G)$ is the smallest value among all Hamiltonian colorings of G .

EX: Figure.1 (a) shows a graph H of order 5. A vertex coloring c of H is shown in Figure.1 (b). Since $D(u, v) + |c(u) - c(v)| \geq 4$ for every two distinct vertices u and v of H , it follows that c is a Hamiltonian coloring and so $hc(c) = 4$. Hence $hc(H) \geq 4$. Because no two of the vertices t, w, x , and y are connected by a Hamiltonian path, these must be assigned distinct colors and so $hc(H) \geq 4$. Thus $hc(H) = 4$.



1. A graph with Hamiltonian chromatic number 4

If a connected graph G of order n has Hamiltonian chromatic number 1, then $D(u, v) = n - 1$ for every two distinct vertices u and v of G and consequently G is Hamiltonian-connected, that is, every two vertices of G are connected by a Hamiltonian path. Indeed, $hc(G) = 1$ if and only if G is Hamiltonian – connected. Therefore, the Hamiltonian Chromatic Number of a connected graph G can be considered as a measure of how close G is to being Hamiltonian connected. That is the closer $hc(G)$ is to 1, the closer G is to being Hamiltonian connected. The three graphs H_1, H_2 and H_3 shown in below figure 2 are all close (in this sense) to being Hamiltonian-connected since $hc(H_i) = 2$ for $i = 1, 2, 3$.



H_3

2. Three graphs with Hamiltonian chromatic number 2

2 Theorem: For every integer $n \geq 3$, $hc(K_{1,n-1}) = (n-2)^2 + 1$

Proof: Since $hc(K_{1,2}) = 2$ (See H_1 in Figure 2) we may assume that $n \geq 4$.

Let $G = K_{1,n-1}$ where $V(G) = \{v_1, v_2, \dots, v_n\}$ and v_n is the central vertex. Define the coloring c of G by $c(v_n) = 1$ and $C(v_i) = (n-1) + (i-1)(n-3)$ for $1 \leq i \leq n-1$. Then c is a Hamiltonian coloring of G and

$hc(G) \leq hc(c) = c(v_{n-1}) = (n-1) + (n-2)(n-3) = (n-2)^2 + 1$. It remains to show that $hc(G) \geq (n-2)^2 + 1$.

Let c be a Hamiltonian coloring of G such that $hc(c) = hc(G)$. Because G contains no Hamiltonian path, c assigns distinct colors to the vertices of G . We may assume that $C(v_1) < c(v_2) < \dots < c(v_{n-1})$. We now consider three cases, depending on the color assigned to the central vertex v_n .

Case 1.

$c(v_n) = 1$.

Since

$$D(v_1, v_n) = 1 \text{ and } D(v_i, v_{i+1}) = 2 \text{ for } 1 \leq i \leq n-2.$$

It follows that

$$C(v_{n-1}) \geq 1 + (n-2) + (n-2)(n-3) = (n-2)^2 + 1$$

And so

$$hc(G) = hc(c) = c(v_{n-1}) \geq (n-2)^2 + 1.$$

Case 2.

$C(v_n) = hc(c)$

Thus, in this case,

$$1 = c(v_1) < c(v_2) < \dots < c(v_{n-1}) < c(v_n)$$

Hence

$$C(v_n) \geq 1 + (n-2)(n-3) + (n-2) = (n-2)^2 + 1$$

And so

$$hc(G) = hc(c) = c(v_n) \geq (n-2)^2 + 1.$$

Case 3.

$C(v_j) < c(v_n) < c(v_{j+1})$ for some integer j with $1 \leq j \leq n-2$.

Thus $c(v_1) = 1$ and $c(v_{n-1}) = hc(c)$.

in this case

$$C(v_j) \geq 1 + (j-1)(n-3),$$

$$C(v_n) \geq c(v_j) + (n-2)$$

$$C(v_{j+1}) \geq c(v_n) + (n-2) \text{ and}$$

$$C(v_{n-1}) \geq c(v_{j+1}) + [(n-1) - (j+1)](n-3).$$

Therefore,

$$\begin{aligned} C(v_{n-1}) &\geq 1 + (j-1)(n-3) + 2(n-2) + (n-j-2)(n-3) \\ &= (2n-3) + (n-3)^2 = (n-2)^2 + 2 > (n-2)^2 + 1. \end{aligned}$$

$$\text{And so } hc(G) = hc(c) = c(v_{n-1}) > (n-2)^2 + 1.$$

Hence in any case,

$$hc(G) \geq (n-2)^2 + 1 \text{ and so } hc(G) = (n-2)^2 + 1.$$

3 Theorem: For every integer $n \geq 3$, $hc(C_n) = n-2$.

Proof.

Since we noted that $hc(C_n) = n-2$ for $n = 3, 4, 5$. We may assume that $n \geq 6$. Let $C_n = (v_1, v_2, \dots, v_n, v_1)$. Because the vertex coloring c of C_n defined by $c(v_1) = c(v_2) = 1$, $c(v_{n-1}) = c(v_n) = n-2$ and $c(v_i) = i-1$ for $3 \leq i \leq n-2$ is a Hamiltonian coloring, it follows that $hc(C_n) \leq n-2$. Assume, to the contrary, that $hc(C_n) < n-2$ for some integer $n \geq 6$. Then there exists a Hamiltonian $(n-3)$ coloring c of C_n . We consider two cases, according to whether n is odd or n is even.

Case 1.

n is odd: Then $n = 2k + 1$ for some integer $k \geq 3$. Hence there exists a Hamiltonian $(2k-2)$ coloring c of C_n . Let,

$$A = \{1, 2, \dots, k-1\} \text{ and } B = \{k, k+1, \dots, 2k-2\}$$

For every vertex u of C_n , there are two vertices v of C_n such that $D(u, v)$ is minimum (and $d(u, v)$ is maximum), namely $D(u, v) = d(u, v) + 1 = k+1$. For $u = v_i$, these two vertices v are v_{i+k} and v_{i+k+1} (where the addition in $i+k$ and $i+k+1$ is performed modulo n). Since c is a Hamiltonian coloring.

$$D(u, v) + |c(u) - c(v)| \geq n-1 = 2k. \text{ Because } D(u, v) = k+1, \text{ it follows that}$$

$$|c(u) - c(v)| \geq k-1.$$

Therefore, if $c(u) \in A$, then the colors of these two vertices v with this property must belong to B . In particular, if $c(v_i) \in A$, then $c(v_{i+k}) \in B$. Suppose that there are a vertices of C_n whose colors belong to A and b vertices of C_n whose colors belong to B . Then $b \geq a$. However, if $c(v_i) \in B$, then $c(v_{i+k})$ belongs to A implying that $a \geq b$ and so, $a=b$. since $a+b=n$ and n is odd, this is impossible.

Case 2.

n is even: Then $n = 2k$ for some integer $k \geq 3$. Hence there exists a Hamiltonian $(2k-3)$ coloring c of C_n . For every vertex u of C_n , there is a unique vertex v of C_n for which $D(u, v)$ is minimum (and $d(u, v)$ is maximum), namely, $d(u, v) = k$. For $u = v_i$, this vertex v is v_{i+k} (where the addition in $i+k$ is performed modulo n).

Since c is a Hamiltonian coloring, $D(u, v) + |c(u) - c(v)| \geq n-1 = 2k-1$. Because $D(u, v) = k$, it follows that $|c(u) - c(v)| \geq k-1$. This implies, however, that if $c(u) = k-1$, then there is no color that can be assigned to u to satisfy this requirement. Hence no vertex of C_n can be assigned the color $k-1$ by c .

$$\text{Let, } A = \{1, 2, \dots, k-2\} \text{ and } B = \{k, k+1, \dots, 2k-3\}.$$

Thus $|A| = |B| = k-2$. If $c(v_i) \in A$, then $c(v_{i+k}) \in B$. Also, if $c(v_i) \in B$, then $c(v_{i+k}) \in A$. Hence there are k vertices of C_n assigned colors from B . Consider two adjacent vertices of C_n , one of which is assigned a color from A and the other is assigned a color from B . We may assume that $c(v_1) \in A$ and $c(v_2) \in B$. Then $c(v_{k+1}) \in B$. Since $D(v_2, v_{k+1}) = k+1$, it follows that $|c(v_2) - c(v_{k+1})| \geq k-2$. Because $c(v_2), c(v_{k+1}) \in B$, this implies that one of $c(v_2)$ and $c(v_{k+1})$ is at least $2k-2$. This is a contradiction.

3.1 Proposition: If H is a spanning connected subgraph of a graph G , then

$$hc(G) \leq hc(H)$$

Proof.

Suppose that the order of H is n . Let c be a Hamiltonian coloring of H such that $hc(c) = hc(H)$. Then $D_H(u, v) + |c(u) - c(v)| \geq n-1$ for every two distinct vertices u and v of H . since $D_G(u, v) \geq D_H(u, v)$ for every two distinct vertices u and v of H , it follows that $D_G(u, v) + |c(u) - c(v)| \geq n-1$ and so c is a Hamiltonian coloring of G as well. Hence $hc(G) \leq hc(c) = hc(H)$.

3.2 Proposition: Let H be a Hamiltonian graph of order $n-1 \geq 3$. If G is a graph obtained by adding a pendant edge to H , then $hc(G) = n-1$.

Proof. Suppose that $C = (v_1, v_2, \dots, v_{n-1}, v_1)$ is a Hamiltonian cycle of H and $v_1 v_n$ is the pendant edge of G . Let c be a Hamiltonian coloring of G . Since $D_G(u, v) \leq n-2$ for every two distinct vertices u and v of C , no two vertices

of C can be assigned the same color by c . Consequently,
 $hc(c) > n - 1$ and so $hc(G) \geq n - 1$.
 Now define a coloring c' of G by

$$C^1(v_i) = \begin{cases} i & \text{if } 1 < i < n-1 \\ n-1 & \text{if } i = n. \end{cases}$$

We claim that c' is a Hamiltonian coloring of G . First let
 v_j and v_k be two vertices of C where $1 \leq j < k \leq n-1$.
 The $|c^1(v_j) - c^1(v_k)| = k - j$ and

$$D(v_j, v_k) = \max \{k-j, (n-1) - (k-j)\}.$$

In either case, $D(v_j, v_k) \geq n-1 + j - k$ and so

$$D(v_j, v_k) + |c^1(v_j) - c^1(v_k)| \geq n-1.$$

For $1 \leq j \leq n-1$, $|c^1(v_j) - c^1(v_n)| = n-1-j$, while

$$D(v_j, v_n) \geq \max \{j, n-j+1\}$$

And so, $D(v_j, v_n) \geq j$.

Therefore,

$$D(v_j, v_n) + |c^1(v_j) - c^1(v_n)| \geq n-1.$$

Hence, as claimed, c' is a Hamiltonian coloring of G and
 so

$$hc(G) \leq hc(c') = c^1(v_n) = n-1.$$

4 Theorem: for every connected graph G of order $n \geq 2$,

$$hc(G) \leq (n-2)^2 + 1.$$

Proof. First, if G contains a vertex of degree $n-1$, then G
 contains the star $K_{1,n-1}$ as a spanning sub graph. Since hc
 $(K_{1,n-1}) = (n-2)^2 + 1$ it follows by proposition 1 that $hc(G)$
 $\leq (n-2)^2 + 1$. Hence we may assume that G contains a
 spanning tree T that is not a star and so its complement T
 contains a Hamiltonian path $P = (v_1, v_2, \dots, v_n)$. Thus $v_i v_{i+1}$
 $\notin E(T)$ for $1 \leq i \leq n-1$ and so $D_T(v_i, v_{i+1}) \geq 2$. Define a
 vertex coloring c of T by

$$C(v_i) = (n-2) + (i-2)(n-3) \text{ for } 1 \leq i \leq n.$$

Hence

$$hc(c) = c(v_n) = (n-2) + (n-2)(n-3) = (n-2)^2$$

Therefore, for integers i and j with $1 \leq i < j \leq n$,

$$|c(v_i) - c(v_j)| = (j-i)(n-3).$$

If $j = i+1$, then

$$D(v_i, v_j) + |c(v_i) - c(v_j)| \geq 1 + 2(n-3) = 2n-5 \geq n-1.$$

Thus c is a Hamiltonian coloring of T . therefore,

$$hc(G) \leq hc(T) \leq hc(c) = c(v_n) = (n-2)^2 < (n-2)^2 + 1,$$

Which completes the proof

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REFERENCES;

- [1].G.Chartrand, L.Nebesky, and P.Zhang, Hamiltonian
 graphs. Discrete Appl.Math. 146(2005) 257-272.
- [2].O.Ore Note on Hamalton circuits. Amer. Math.
 Monthly 67(1960)
- [3]. P.G Tait, Remarks on the colorings of maps
 proc.Royal soc.Edinburgh 10 (1880)
- [4]. H.Whitney, the Colorings of Graphs.Ann.Math.33
 (1932) 688-718.
- [5]. D.R.Woodall, List colourings of graphs. Survey in
 combinatories 2001 Cambridge univ. press, Cambridge,
 (2001) 269-301.