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A NOTE ON COMPLETELY SEMI-PRIME IDEALS IN NEAR-RINGS

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Abstract

The aim of the present paper is to prove that (i) Every completely semi-prime ideal I of N is the intersection of all completely prime ideals of I; and (ii) If I is a completely semi-prime ideal of N, then an ideal P of N is a minimal completely prime ideal of I if and only if P is a minimal prime ideal of I, where N is a zero-symmetric near-ring. We also discussed the relations between the prime radical and completely prime radical. In particular, we proved that these two radicals are equal in case of a particular type of near-rings called PC near-rings. Related examples were presented whenever necessary.

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Key Words: Near-ring, prime ideal, minimal prime ideal, completely prime ideal, semi-prime ideal, completely semi-prime ideal.

1. Introduction
In this section, we present some fundamental definitions and results from the literature which are used in the sequel. An algebraic system (N, +, **.**) is called a near-ring if it satisfies the following three conditions: (i) (N, +) is a group; (ii) (N, **.**) is a semi-group; (iii) $(x + y)z = xz + yz$ for all $x, y, z \in N$. Further if N satisfies the condition that "$x0 = 0$ for all $x \in N$", where '0' is the identity element of (N, +), then N is called a zero-symmetric near-ring. We abbreviate (N, +, **.**) by N. Throughout N stands for a zero-symmetric near-ring. If $A_1, A_2, \ldots, A_n$ are subsets of N, then the set $\{a_1a_2\ldots a_n / a_i \in A_i$ for $1 \leq i \leq n\}$ is denoted by $A_1A_2\ldots A_n$. In the special case when all A_i's are equal to the same set A, we denote $A_1A_2\ldots A_n$ by A^n. If $A = \{a\}$, then we write aB for AB.

If A is a subset of N, then the ideal generated by A, that is, the intersection of all ideals of N containing A, is denoted by <A>. For any $x \in N$, we write <x> instead of <{x}>. If A is a subset of N, then the set $\{n \in N / n \notin A\}$ is denoted by N \ A. An ideal P of N is called a prime ideal if for any two ideals J and I of N, $IJ \subseteq P$, implies either $I \subseteq P$ or $J \subseteq P$. An ideal S of N is said to be semi-prime if for any ideal I of N, $I^2 \subseteq S$, implies $I \subseteq S$. An ideal P of N is said to be a completely prime ideal if for $a, b \in N$, $ab \in P$, implies either $a \in P$ or $b \in P$. An ideal S of N is said to be a completely semi-prime ideal if for any element $a \in N$, $a^2 \in S$, implies $a \in S$.

A near-ring N is said to be a prime (completely prime, semi-prime, completely semi-prime, respectively) near-ring if (0) is a prime (completely prime, semi-prime, completely semi-prime, respectively) ideal. The intersection of all prime (completely prime, respectively) ideals of N is called the prime (completely prime, respectively) radical of N and it is denoted by P-rad(N) (C-rad(N), respectively). For any proper ideal I of N, the intersection of all prime (completely prime, respectively) ideals of N containing I, is called the prime (completely prime, respectively) radical of I and is denoted by P-rad(I) (C-rad(I), respectively).

Prime ideals, semi-prime ideals and prime radical were studied by several authors in [1], [3] to [12], and [14].

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(Van der Walt [14]) A subset M of N is called an m-system if for all a, b $\in$ M, we have $\langle a\rangle\langle b\rangle \cap M \neq \phi$. A subset S of N is called an sp-system if for all a $\in$ S, $\langle a\rangle^2 \cap S \neq \phi$. *(Sambasiva Rao and Satyanarayana [7])* A subset M of N is called an m^*-system if for all a, b $\in$ M, we have $a\langle b\rangle \cap M \neq \phi$ (that is, there exists an element $b_1 \in \langle b\rangle$ such that $ab_1 \in M$). A subset S of N is called an sp^*-system if for all a $\in$ S, we have $a\langle a\rangle \cap S \neq \phi$ (that is, there exists an element a_1 in $\langle a\rangle$ such that $aa_1 \in S$). Now it is easy to verify that (i) An m^*-system is an m-system and an sp^*-system is an sp-system; and (ii) Every m^*-system is an sp^*-system, but the converse is not true. As an example, if N = $\mathbb{Z}$ is the ring of integers, then N \ <10> is an sp^*-system, but not m^*-system. A straight forward verification provides the following.

1.1 Remark: Let P be an ideal of N. Then

(i) P is prime if and only if for all a, b $\in$ N $a\langle b\rangle \subseteq P$, implies either a $\in$ P or b $\in$ P.

(ii) P is semi-prime if and only if a $\in$ N, $a\langle a\rangle \subseteq P$, implies a $\in$ P.

(iii) P is prime if and only if N \ P is an m^*-system.

(iv) P is semi-prime if and only if N \ P is an sp^*-system.

1.2 ***Theorem*** *(Sambasiva Rao & Satyanarayana [7])*: Let I be a proper ideal of N. Then

P-rad(I) = {x $\in$ N / every m^*-system that contains 'x' also contains an element of I}.

1.3 ***Lemma*** *(Sambasiva Rao & Satyanarayana [7])*: Let S be an sp^*-system and a $\in$ S. Then there exists an m^*-system M such that $a \in M \subseteq S$.

Let I be an ideal of N. Then a prime (completely prime, respectively) ideal of N containing I is called a minimal prime (minimal completely prime, respectively) ideal of I if P is minimal in the set of all prime (completely prime, respectively) ideals containing I.

1.4. ***Theorem*** *(Van der Walt [14], Sambasiva Rao [6] & Groenewald Nico [1])*: Let I be a semi-prime ideal of a near-ring N. Then I is the intersection of all minimal prime ideals of I.

1.5 ***Theorem*** *(Pilz [2] or Sambasiva Rao [6])*: Let I be an ideal of N and M a nonempty m-system in N such that $I \cap M = \phi$. Then

(a) There exists an m-system M^* which is maximal relative to the properties $M \subseteq M^*$ and $I \cap M^* = \phi$.

(b) If M^* is an m-system maximal relative to the properties $M \subseteq M^*$ and $I \cap M^* = \phi$, then N \ M^* is a minimal prime ideal of I.

1.6. ***Theorem*** *(Page 71, Theorem 2.105 (b), Pilz [2])*: Prime radical of a near-ring is a nil ideal.

1.7. ***Theorem*** *(Page 90, Theorem 3.40, Pilz [2], Ramakotaiah [3])*: If N has DCC on N-subgroups and M is an N-subgroup of N, then M is nilpotent $\Leftrightarrow$ M is nil.

For other necessary preliminary definitions & results we refer [2] and [13].

2. Completely Semi-prime Ideals

We start this section with the following lemma.

2.1. ***Lemma***: (i) If I is a completely semi-prime ideal of a near-ring N, then for any a, b $\in$ N, ab $\in$ I $\Rightarrow$ ba $\in$ I.

(ii) If a $\in$ N and I is a semi prime ideal of N, then (I:a) = {x $\in$ N / xa $\in$ I} is an ideal of N.

Proof: (i) Suppose a, b $\in$ N such that ab $\in$ I. Then $(ba)^2 = (ba)(ba) = b(ab)a \in I$. Since I is a completely semi-prime ideal, we have that ba $\in$ I.

(ii) It is obvious that (I:a) is a left ideal. We now show that (I:a) is a right ideal of N. Let x $\in$ (I:a) and n $\in$ N. Since x $\in$ (I:a), we have that xa $\in$ I. By (i), ax $\in$ I. Since I is a left ideal nax $\in$ I and by using (i) again, xna $\in$ I. Thus (I:a) is a right ideal. The proof is completed.

2.2. ***Theorem***: (a) If S is a semi-prime ideal of N, then the following are equivalent:

(i) If $x^2 \in S$, then $<x>^2 \subseteq S$.

(ii) S is completely semi prime

(iii) If $xy \in S$, then $<x><y> \subseteq S$.

(b) An ideal P of N is prime and completely semi-prime if and only if it is completely prime.

Proof: (a): (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii): Let $x, y \in N$ such that $xy \in S$. By Lemma 2.1(ii), (I:y) is an ideal. So $<x> \subseteq (I:y)$, which implies $<x>y \subseteq I$. By Lemma 2.1 (i), $y<x> \subseteq I$. Again by using the same logic we get $<y> <x> \subseteq I$ and so we have $<x><y> \subseteq I$.

(iii) $\Rightarrow$ (i) is clear.

(b) follows from (a).

2.3. ***Notation***: we write

L = the set of all nilpotent elements of N.

sn(N) = the sum of all nilpotent ideals of N.

The following remark is obvious.

2.4. ***Remark***: (i) P-rad (N) $\subseteq$ C-rad(N).

(ii) $sn(N) \subseteq L \subseteq S$, for any completely semi-prime ideal S of N.

Before proving our main theorem, we need the following three lemmas.

2.5. ***Lemma***: If M is an m-system and I is a completely semi-prime ideal of N such that $M \cap I = \phi$, then the sub-semi-group of (N, .) generated by M is contained in $N \setminus I = \{n \in N / n \notin I\}$.

Proof: It suffices to show that if $a_1, a_2, \ldots, a_k \in M$, then $a_1a_2a_3\ldots a_k \in N\backslash I$. Let $a_1, a_2, \ldots, a_k \in M$. Since $a_1, a_2 \in M$ and M is an m-system, there exists $a_2^1 \in <a_1>$ and $a_2^* \in <a_2>$ such that $a_2^1a_2^* \in M$. Since $a_2^1a_2^* \in M$ and $a_3 \in M$, there exists $a_3^1 \in <a_2^1a_2^*>$, $a_3^* \in <a_3>$ such that $a_3^1a_3^* \in M$. If we continue this up to k steps, we have elements $a_2^1, a_3^1, \ldots, a_k^1, a_2^*, a_3^*, \ldots, a_k^*$ in N such that $a_i^1a_i^* \in M$ for $2 \le i \le k$ and $a_i^1 \in <a^1_{i-1}a^*_{i-1}>$, $a_i^* \in <a_i>$.

If $a_1a_2\ldots a_k \notin (N \setminus I)$, then $a_1a_2\ldots a_k \in I$. Now $a_1a_2\ldots a_k \in I \Rightarrow a_2^1a_2a_3 \ldots a_k \in I$ (by Lemma 2.1 (ii)) $\Rightarrow a_2a_3 \ldots a_ka_2^1 \in I$ (by Lemma 2.1 (i)) $\Rightarrow a_2^*a_3\ldots a_ka_2^1 \in I \Rightarrow a_2^1a_2^*a_3\ldots a_k \in I \Rightarrow a_3^1a_3\ldots a_k \in I$. If we continue this process up to k steps, we get $a_k^1a_k^* \in I$. Hence $a_k^1a_k^* \in M \cap I = \phi$, a contradiction. Therefore $a_1, a_2, \ldots, a_n \in N\backslash I$.

2.6. ***Lemma***: If P is a minimal prime ideal of I, then N\P is an m-system maximal relative to the property that $(N\backslash P) \cap I = \phi$.

Proof: If M = N\P is not a maximal m-system relative to the property $M \cap I = \phi$, then by Theorem 1.5, there exists an m-system M^* maximal relative to the properties $M \subseteq M^*$ an $M^* \cap I = \phi$. Again by Theorem 1.5, $N\backslash M^*$ is minimal prime ideal of I in N. Now $N\backslash P = M \subseteq M^*$ and hence $P = (N\backslash M) \supseteq (N\backslash M^*)$. Since $(N\backslash M^*)$ is minimal prime ideal of I, we have that $P = N\backslash M^*$. Now it follows that $M = M^*$. The proof is completed.

2.7. ***Lemma***: Let P be a minimal prime ideal of a completely semi-prime ideal I. Then P is a completely prime ideal. Moreover, P is a minimal completely prime ideal of I.

Proof: Suppose $P \neq N$. Then N\P is an m-system such that $M \cap I = \phi$. By Lemma 2.5, the multiplicative sub semi group S generated by M is contained in N\I. Since P is a minimal prime ideal of I, by Lemma 2.6, M is a maximal m-system relative to "having empty intersection with I". Since S is a sub-semi-group it is an m-system and hence M = S. Since S = (N\P) is sub-semi-group of (N, .), we have that P is a completely prime ideal containing I.

Now we are ready to prove our main theorem.

2.8. ***Theorem***: Let I be a completely semi-prime ideal of N. Then I is the intersection of all minimal completely prime ideals of I.

Proof: If I is a completely semi-prime ideal of N, then I is a semi-prime ideal. By Theorem 1.4, I is the intersection of all minimal prime ideals of I. Since I is a completely semi-prime, by Lemma 2.7., every minimal prime ideal of I is minimal completely prime ideal of I and so I is the intersection of some collection of minimal completely prime ideals of I. The proof is completed.

2.9. ***Theorem***: If P is a prime ideal and I is a completely semi-prime ideal, then P is a minimal prime of I if and only if P is a minimal completely prime of I.

Proof: Suppose P is minimal completely prime ideal of I. If P is not minimal prime ideal of I, then there exists a prime ideal P^1 containing I and $P \neq P^1$. Now $\wp$ = {J / J is a prime ideal such that $P \supseteq J \supset I$} is non-empty and by using Zorn's lemma, we get a minimal element in $\wp$, say P^*. Now P^* is a minimal prime of I. Now by Lemma 2.7, P^* is completely prime of I. Since $P \supseteq P^* \supset I$ and P is a minimal completely prime of I, we have $P = P^*$ and so P is minimal prime, a contradiction. The rest follows from Lemma 2.7.

2.10. ***Corollary***: If I is a completely semi-prime ideal of N, then I is the intersection of all completely prime ideals of N containing I.

The following example shows that Theorem 2.8, fails if N is not a zero symmetric near-ring.

2.11. ***Example***: Let (G, +) be the Klein four group and G = {0, a, b, c}. Define multiplication on G as follows:

.	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	a	b	c
c	a	0	c	b

Now (G, +, .) is a near-ring which is not zero symmetric since $a0 = a \neq 0$. In this near-ring {0, a} is only the non-trivial ideal and also it is completely prime. The ideal (0) is completely semi-prime, but not completely prime (since c.a = 0 and $a \neq 0 \neq c$). Hence the completely semi-prime ideal (0) can not be written as the intersection of its minimal completely prime ideals.

3. Relations between P-rad(N), C-rad(N) and L

In this section, we discuss some interesting consequences. The following example shows that even in case of rings the sets: P-rad(N), C-rad(N) and L are need not be equal.

3.1. ***Example***: Let N be the set of all 3 x 3 matrices over the ring of integers. If

$x = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$, $y = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, then x and y are nil-potent elements of N (since $x^4 = y^4 = 0$).

Consider $x + y = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. The element $(x + y)^2$ $= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ is not nilpotent. So L can not be an ideal. We can conclude that P-rad(N) $\neq$ L $\neq$ C-rad(N). By Theorem 1.6, P-rad(N) $\subseteq$ L. Since C-rad(N) is completely semi-prime, every nilpotent element belongs to C-rad(N) and so L $\subseteq$ C-rad(N). Now we get that

P-rad(N) $\subsetneq$ L $\subsetneq$ C-rad(N).

3.2. ***Theorem***: If N is a near-ring such that P-rad(N) is completely semi-prime, then P-rad(N) = L = C-rad(N).

Proof: Since P-rad(N) is completely semi-prime, by Theorem 2.8, P-rad(N) = $\cap$ P, P ranges over all completely semi-prime ideals containing P-rad(N). This implies that P-rad(N) $\supseteq$ C-rad(N). Already we have P-rad(N) $\subseteq$ L $\subseteq$ C-rad(N). Thus P-rad(N) = L = C-rad(N).

3.3. ***Theorem***: If N has DCC on N-subgroups, then P-rad(N) is completely semi-prime if and only if L is an ideal. In this case P-rad(N) = L = C-rad (N).

Proof: If L is an ideal, then by Theorem 1.7, L is nilpotent. Since P-rad(N) is a semi-prime ideal, we have that L $\subseteq$ P-rad(N). Since L is the set of all nilpotent

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elements, L is completely semi-prime. By Theorem 2.8, L = C-rad(N). So L = P-rad(N) = C-rad(N), which shows that P-rad(N) is a completely semi-prime ideal. Converse follows from Theorem 3.2.

4. PC Near-rings

In this section, we discuss the PC near-rings.

4.1 Definition: A near-ring N is said to be pseudo-commutative (PC, in short) near-ring if $x_1, x_2, \ldots, x_k \in N$ such that $x_1 x_2 \ldots x_k = 0$, implies $<x_1><x_2> \ldots <x_k> = 0$.

4.2 Theorem: If N is a PC near-ring, then sn(L) = L = P-rad(N).

Proof: Let $x \in L$. Then there exists an integer k such that $x^k = 0$. Since N is a PC near-ring $<x>^k = 0$, which shows that $x \in$ sn(N). Therefore L $\subseteq$ sn(N) $\subseteq$ P-rad(N) $\subseteq$ L, which implies L= sn(N) = P-rad(N).

As a consequence of Theorem 2.8., we have the following:

4.3. ***Corollary***: (Ramakotaiah [4]): Let N be a PC near-ring. Then L is the intersection of all minimal completely prime ideals of N. Hence the set of all nilpotent elements is an ideal.

4.4. ***Corollary***: If N is a PC near-ring, then

sn(N) = P-rad(N) = L = C-rad(N).

The following example shows that if N is not a PC near-ring, then the Corollary 4.3, need not be true.

4.5. ***Example***: Consider Z_5, the near-ring of integers modulo 5. Define multiplication '*' on Z_5 as follows.

*	0	1	2	3	4
0	0	0	0	0	0
1	0	0	4	1	0
2	0	0	3	2	0
3	0	0	2	3	0
4	0	0	1	4	0

Now $(Z_5, +, .)$ is a near-ring. Z_5 has no proper ideals and it is not PC near-ring (since $4^2 = 0$ but $<4><4> = z_5^2$ and $z_5^2 \neq (0)$). L does not form an ideal (since $1 \in$ L and $3 \notin$ L). Hence P-rad(z_5) = (0) $\neq$ L $\neq$ Z_5 = C-rad(Z_5).

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