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CONFIGURATIONS, PROJECTIONS AND FREE COMPLETIONS ON UNIQUELY PLANAR COLOURED GRAPHS

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Abstract
In this paper configurations, projections and free completions on uniquely planar coloured graphs.

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Introduction:

1.1 Combinatorial Representations of Drawings

In this section we develop a combinatorial description of drawings. Before we do this, one minor detail must be disposed of. A consequence of the definition of drawing is that an edge incident to a vertex v can spiral around v an infinite number of times. This and other types of refractory edges can make the analysis of drawings complicated. Happily, this problem can be circumvented without loss of generality through the following lemma, a proof of which appears in Bollobas

Lemma 1.1.1 *Let G be a drawing of a graph. There is a drawing of G in which every edge of G is piecewise polygonal.*

The advantage gained by this assumption will be made clear by the next lemma and some definitions which follow. We will then prove two lemma's which lay the foundation for a combinatorial description of drawings. Each of these lemmas relies on the piecewise polygonal drawing of edges.

Lemma 1.1.2 *Let G be a piecewise polygonal drawing of a graph. There is an $\varepsilon > 0$ such that if $v \in V(G)$ and $g_1, g_2, \ldots, g_{dG(v)}$ are the edges of G that are incident to v then.*

1) $\overline{D(v,\varepsilon)} \cap (V(G) \cup (E(G) - \{g_1, \ldots, g_{dG(v)}\})) = \{v\}$.

2) $\overline{D(v,\varepsilon)} \cap g_i$ is a straight line for every $1 \le i \le d_G(v)$.

The proof of this lemma will be omitted.

For a given piecewise polygonal drawing of G, we call any ε which satisfies the conditions 1) and 2) of Lemma 1.1.2 a critical radius for G. Henceforth, we will assume that all drawings of graphs are piecewise polygonal. Let G be a drawing, let ε be a critical radius for G and let $v \in V(G)$. Consider all points of intersection of the circle $C(v, \varepsilon)$ with the edges g1, . . . , $g_{dG(v)}$ that are incident to v. Because of property 1), each edge in E(G) that is incident to v intersects $C(v, \varepsilon)$ at least once. From property 2), each edge incident to v intersects $C(v, \varepsilon)$ at most once. Denote by xi the unique point in C(v, ε) $\bigcap$ g_i. We will say that x_j follows xi on C(v, ε) (or if C is evident from the context simply that x_j follows xi if A(x_i, x_j) $\bigcap$ {x_1, . . . , x_d(v)} = {x_i, x_j}. If {j_1, j_2, . . . , j_d(v)} = {1, 2, . . . , d(x)} is such that $x_{j\,i+1}$ follows x_{ji} for every 1 $\leq$ i $\leq$ d(v) (where if i = $d_{(x)}$, j_{i+1} is interpreted to be j_1), then g_{j1} , g_{j2} , . . . , $g_{\,jd(x)}$ is a clockwise listing in G at C of edges incident to v. It is clear that any cyclic shift of j_1, . . . , $j_{d(x)}$ naturally gives rise to another clockwise listing at C of edges incident to v. Property 2) of Lemma 1.1.2 implies that any two critical radii will lead to the same clockwise listing in G of edges incident to v. It is also natural to say that a set of vertices u_1, u_2, . . . , $u_{dG(v)}$ is a clockwise listing in G of neighbors of v if g_1, . . . , $g_{dG(v)}$ is a clockwise listing of edges in G having the property that the endpoint in V (G) - v of gi is ui for 1 $\leq$ i $\leq$ d_G(v). The next lemma gives the basic but fundamental connection of clockwise listings to drawings.

Lemma 1.1.3 *Suppose that G is a near-triangulation and that v $\in$ V (G). Suppose that g_1, . . . , $g_{\,dG(v)}$ is a clockwise listing in G of edges incident to v. For every 1 $\leq$ i $\leq$ d_G(v), there is a face ri of G which is incident to both g_i and g_{i+1} (where if i = d_G(v) we interpret g_{i+1} as g_1). Moreover, if d_G(v) $\leq$ 3, and r_i is finite and triangular then ri is the unique finite face of G that is incident to g_i and g_{i+1} for 1 $\leq$ i $\leq$ d_G(v).*

Proof: Let ε be a critical radius of G, and for 1 $\leq$ i $\leq$ d_G(v), let xi denote the unique point of intersection of the edge g_i with C(v, ε). It follows that A(x_i, x_{i+1}, v, ε) = A(x_i, x_{i+1}) satisfies A(x_i, x_{i+1}) $\bigcap$ {x_1, . . . , xd_G(v)} = {x_i, x_{i+1}}, for every 1 $\leq$ i $\leq$ d_G(v) (with the usual convention that when i = d_G(v), i + 1 is interpreted to be 1). From property 2) of Lemma 1.1.2, we may also conclude that A(x_i, x_{i+1}) $\bigcap$ G = {x_i, x_{i+1}} for 1 $\leq$ i $\leq$ d_G(v). Therefore A(x_i, x_{i+1}) − {x_i, xi_{+1}} is a subset of r_i, and this implies that $g_i \subset \bar{r}_i$ and $g_{i+1} \subset \bar{r}_i$ for 1 $\leq$ i $\leq$ d_G(v). This completes the first part of the proof.

Now assume that d_G(v) $\geq$ 3 and that r_i is a triangular face. Suppose that r $\neq$ $r_i$is another face of G that is incident to both g_i and g_{i+1}. By relabeling we may assume that i = 1. The face r_2 is incident to the edges g_2 and g_3. The edge g_2 is incident to at most two faces and therefore r_2 $\in$ {r, r_1}. If r_2 = r_1, then r_1 is incident to the three edges g_1, g_2, g_3 which is impossible since these three edges are all incident to v and r_1 is a triangular face. Therefore r_2 = r. However r is incident to both of g_1 and g_2 and since r = r_2, r must also incident to g_3 and this implies that r is not triangular and hence is the infinite face of G. This completes the proof of Lemma 1.1.3.

1.2 Configurations, Free Completions, and Projections

The next definition is taken from Robertson et. al. A configuration K is a near triangulation G = G(K) together with a function K defined on the vertex set of G(K) such that

(i) For every vertex v $\in$ V (G), G(K)-v has at most two components, and if there are two, then γ_K(v) = d_G(v) + 2.

(ii) For every vertex v $\in$ V (G), if v is not incident with the infinite region, then γ_K(v) = $d_{\ G(v)}$ otherwise, γ_K(v) > d_G(v) and in either case γ_K(v) $\geq$ 5.

(iii) K has ring size $\geq$ 2, where the ring-size of K is defined to be Σ (γ_K(v) - d_G(v) - 1), summed over all the vertices v of G(K) incident with the infinite region such that G(K) - v is connected.

It will be convenient to represent configurations pictorially, by simply drawing thenear-triangulation in the plane such that all of the finite regions are triangles and where γ(v) is represented by having differing shapes for the vertex v according to the table in figure 1.1

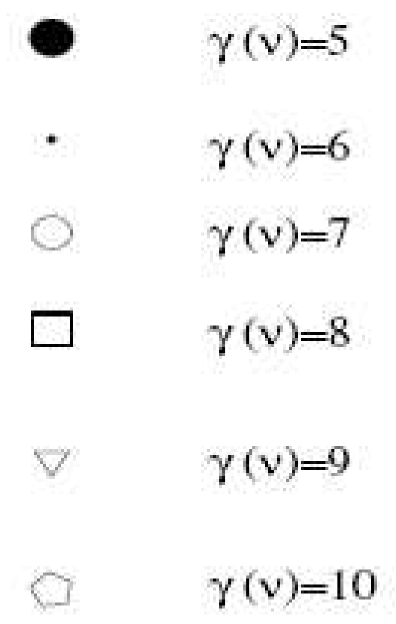

Figure 1.1 The meaning of Vertex Shapes

The set of configurations which we will be most concerned with are the 630 con-figurations that appear in Appendix 1. The configuration in row y and column z of page x of Appendix 1 will be referred to by conf(x, y, z).

If T is a triangulation or a near triangulation, then a configuration K appears inT if G(K) is an induced subdrawing of T, every finite face of G is a face of T and $\gamma_K(v) = d_T(v)$ for every vertex $v \in V(G(K))$.

Two configurations K and L are said to be isomorphic if there is a homeomorphism ϕ of the sphere Σ mapping G(K) to G(L) and a function $\psi : \{5, 6, . . .\} \rightarrow \{5, 6, . . .\}$ such that $\psi(\gamma_K(v)) = \gamma_L(\phi(v))$ for every $v \in V(G)$.

Let K be a configuration. The following definition appears in [98]. A free completion S of K with ring R is a near triangulation such that

(i) The infinite face of S is bounded by an induced circuit R called the ring.

(ii) The graph G(K) is an induced subdrawing of S, G(K) = S - R, every finite region of G(K) is a finite region of S, and the infinite region of G(K) contains the infinite region of S.

(iii) $\gamma_K(v) = d_S(v)$ for every $v \in V(G(K))$.

We will show in Section 1.1 that this free completion is essentially unique.

Suppose a configuration K with underlying graph G = G(K) appears in a triangulation or near-triangulation T. Let A be the set of vertices of V (T) that are eitherin V (G) or are incident to a vertex in V (G). As mentioned above, a configuration could appear in a triangulation or near triangulation in various ways. Thus, while it is conceivable that a configuration K appears in a triangulation T in such a way that the subgraph of T that is induced by the vertices of A is a free completion of K, it is not the case that is has to be this way. One thing that can happen is that vertices on the ring of the free completion can be identified in T. Thankfully, it can be shown that the variety of difierent ways in which a configuration can appear in a triangulation is under control, in the sense that it will sufice to analyze only the free completion. A key part of how this is established is by establishing a close relationship between a free completion S of K, and any triangulation T in which K appears. We now make this more precise, using a definition from [98].

Suppose that T is a triangulation and that S is the free completion of a configuration K with ring R. Let F(S) be the set of finite faces of S. A map ϕ from the setV $(S) \cup E(S) \cup F(S)$ into $V(T) \cup E(T) \cup F(T)$ is called a projection of S into T if

(i) ϕ maps V (S) into V (T), E(S) into E(T) and F(S) into F(T)

(ii) for distinct $u, v \in V(S)$, $\phi(u) = \phi(v)$ only if u, v are both incident with the infinite region of S, for distinct $e, f \in E(S)$, $\phi(e) = \phi(f)$ only if e,f are both incident with the infinite region of S, and for distinct r,s 2 F(S), $\phi(r) \neq \phi(s)$, and

(iii) for $x,y \in V(S) \cup E(S) \cup F(S)$, if x,y are incident in S, then $\phi(x)$, $\phi(y)$ are incident in T.

The next theorem gives some basic properties of free completions. The proof is omitted.

Lemma 1.2.1 *Let S be a free completion of a configuration K with ring R. Then the following are true.*

1) The configuration K appears in S.

2) Every edge e $\in$ E(S) - E(R) is incident to two finite faces of S and has at least one endpoint in V (G).

3) For every v $\in$ V (R) there is an x $\in$ V (G) such that x is adjacent to v.

1) Every finite face of F(S) is incident to some vertex x $\in$ V (G).

5) No v $\in$ V (G) is incident to the infinite face of S.

6) For every v $\in$ V (R), the clockwise listing in S of the edges incident to v is of the form r_1, g_1, . . . , g_p, r_2, where p > 0, r_1, r_2 $\in$ E(R) and g_1, . . . , $g_p \in$ E(S) - E(R).

7) No finite face of S is incident to two edges of E(R).

1.3 The Existence of Projections

Let S be a graph containing a vertex v $\in$ V (S). A portion of a clockwise listing in S of edges incident to v in S is a sequence of at least two edges f_1, f_2, . . . , f_p such that f_1, . . . , f_p, t_1, . . . , t_k is a clockwise listing in S of edges incident to v.

Lemma 1.3.1 *Let G be a near triangulation that is an induced sub drawing of a near triangulation S such that every finite face of G is a finite face of S. Suppose also that $d_S(v) \geq 5$ for v $\in$ V (G) and also that no edge of S which is incident to a vertex of V (G) is incident with the infinite face of S. Let v $\in$ V (G(K)), and let g_1, g_2, . . . , g_j be a portion of the clockwise listing in G of edges incident to v having the property that for every $1 \leq k < j$, the edges g_k and g_{k+1} are incident to a common triangular face of G. Then g_1, . . . , g_j is also a portion of the clockwise listing in S of edges incident to v.*

Proof: Let s_1, s_2, . . . , s_d(v) be a clockwise listing in S of edges incident to v in S. For asuitably small ε there is a circle C(v, ε) having the property that $\overline{D(v,\varepsilon)} \cap$ V (S) = {v}and where $\overline{D(v,\varepsilon)} \cap$ E(S) = $\bigcup_{i=1}^{i=ds(v)}$ s_i[v, xi], where for each $1 \leq i \leq d_S(v)$, $s_i[v, x_i]$ consists of a straight line segment between the vertex v and the final point x_i of s_i on C(v, ε). For x, y $\in \Sigma \cap$ C(v, ε), let A(x, y) be the clockwise arc in C(v, ε) joining x to y.

By the choice of ε, A(x_i, x_{i+1}) $\cap$ (V (S) $\cup$ E(S)) = {x_i, x_{i+1}} for every $1 \leq i \leq d_S(v)$ where for i = $d_S(v)$ we interpret s_{i+1} as s_1).

Since G is a subdrawing of S, E(G) $\subset$ E(S) and so there is a function f : {1, . . . , j} $\rightarrow$ {1, . . . , $d_S(v)$}, such that $g_i = s_{f(i)}$ for i = 1, . . . , j. This also implies that the final point of g_i on C(v, ε) is xf(i) for $1 \leq i \leq j$. Let fi be a finite face of G that is incident to both of the edges gi and g_{i+1}. By Lemma 1.1.3, this finite face fi of G that is incident to both g_i and g_{i+1} is unique. Since G appears in S, every finite face of G is a finite face of S and in particular fi is a finite face of S for $1 \leq i < j$. Now the the open arc A($x_{f(i)}$, $x_{f(i+1)}$) – {$x_{f(i)}$, $x_{f(i+1)}$} of C(v, ε), is contained in f for every $1 \leq i < j$, and so (A($x_{f(i)}$, $x_{f(i+1)}$) – {$x_{f(i)}$, $x_{f(i+1)}$}) $\cap$ (V (S) $\cup$ E(S)) = ϕ, which implies that g_i follows g_{i+1} in the clockwise listing in S of edges incident to v. This establishes Lemma 1.3.1.

Lemma 1.3.2 *Let K be a configuration with underlying graph G, and let v $\in$ V (G). Let g_1, . . . , $g_{dG(v)}$) be a clockwise listing in G of edges incident to v. Let p be the number of indices i in {1, 2, . . . , $d_G(v)$} such that g_i and g_{i+1} are not incident to a common finite face of G. (Here, when i = $d_G(v)$ we understand that $g_{i+1} = g_1$).*

a) If v is not incident to the infinite face then p = 0

b) If v is incident to the infinite face of G and if v is not a cut-vertex of G then p = 1.

c) If v is incident to the infinite face of G and if v is a cut vertex of G, then p = 2.

Proof: We prove that G – {v} has max{|L|, 1} components, where L is the set of indices in $\{1, \ldots, d_G(v)\}$ such that g_i and g_{i+1} are not in a common finite face of G.

By Lemma 1.1.3 , g_i and g_{i+1} must be incident to a common face r_i for $1 \leq i \leq d_G(v)$.

Let $v_i \in V(G) - \{v\}$ be the endpoint of g_i for $1 \leq i \leq d_G(v)$.

First, suppose $|L| = 0$ or $|L| = 1$. After a suitable relabeling of $g_1, \ldots, g_{d_G(v)}$, we may assume that each r_i except possibly $r_{d_G(v)}$ is a finite triangular face so that there is an edge in G with endpoints v_i and v_{i+1} for every $1 \leq i < d_G(v)$. Therefore, the sub graph of G induced by the vertices $\{v_1, \ldots, v_{d_G(v)}\}$ contains a walk and therefore is connected. This implies that $G - \{v\}$ is also connected. This proves the case when $|L| = 0$ or $|L| = 1$.

Now suppose that $p = |L| \leq 2$, let ε be a critical radius and let $L = \{k_1, \ldots, k_p\}$,where we may write $1 \leq k_1 < k_2 < \ldots < k_p \leq d_G(v)$. Suppose that x_i is the unique point in g_i T $C(v, \varepsilon)$ and let $1 \leq i \leq p - 1$. Both of the faces r_{ki} and r_{ki+1} are infinite faces of G, and since G is a near-triangulation, $r_{ki} = r_{ki+1}$. Therefore, and for any two points $x, y \in \Sigma$ where $x \in S(v, \varepsilon, x_{ki}, x_{ki+1}) \bigcap r_{ki}$ and $y \in S(v, \varepsilon, x_{ki+1}, x_{ki+1+1}) \bigcap r_{ki+1}$ there is a line $P_i \subset \Sigma - G$ with endpoints x and y. By letting ε approach 0, x and y approach v, and so we see that for $1 \leq i \leq p - 1$ there is a closed curve C_i such that

1) $C_i \bigcap G = \{v\}$

2) Every edge in $\{g_{ki+1}, \ldots, g_{ki+1}\}$ is a subset of the topological interior of the curve C_i, and every edge in $\{g_1, \ldots, g_{dG(v)}\} - \{g_{ki+1}, \ldots, g_{ki+1}\}$ is a subset of the topological exterior of the curve C_i.

Let Q be a v_{ki}-v_{ki+1} path in G. Condition 2 and the Jordan Curve Theorem imply that (treating Q as a subset of Σ rather than a subgraph of G) $Q \bigcap C_i \neq \phi$. By condition 1), this implies that $Q \bigcap C_i = \{v\}$. This implies that v_{ki} and v_{ki+1} are in different components of $G - \{v\}$. Since this holds for $1 \leq i \leq p - 1$, $G - \{v\}$ will have p components, and this establishes that $G - \{v\}$ has $|L|$ components.

If v is a vertex in G that is not incident to the infinite face of G then $|L| = 0$ and the result claimed in the lemma holds. Also, if v is a non-cut vertex of G that is incident to the infinite face, then $|L| \geq 1$ from the very fact that v is incident to the infinite face, and $|L| < 2$ since v is not a cut-vertex of G. Finally, if v is a cut-vertex of G then $|L| \geq 2$ and property (i) of configurations implies that $|L| \geq 2$. This completes the proof of Lemma 1.3.2.

Lemma 1.3.3 *Let K be a configuration with underlying drawing G which appears in S where S is a either a free completion of K or a triangulation, and let $v \in V(G)$ be a non-cut vertex of G. Let $g_1, \ldots, g_{dG(v)}$ be a clockwise listing in G of edges incident to v having the property that if v is incident to the infinite face of G, then the edges g_1 and $g_{dG(v)}$ are also incident to the infinite face of G. Then a clockwise listing in S of edges incident to v is $g_1, g_2, \ldots, g_{dG(v)}, s_1, \ldots, s_p$, where $d_G(v) + p = \gamma K(v)$, $p = 0$ if and only if v is not incident to the infinite face of G, and $s_1, \ldots, s_p$ are edges in $E(S) - E(G)$ having their endpoints in $\{v\}$ S $(V(S) - V(G))$*

Proof: From Lemma 1.3.2, there is either 0 or 1 indices in $\{1, \ldots, d_G(v)\}$ such that g_i and g_{i+1} are incident to the infinite face of G, (where when $i = d_G(v)$ we interpret g_{i+1} as g_1).

By Lemma 1.3.1, $g_1, g_2, \ldots, g_{dG(v)}$ is a portion of the clockwise listing in S of edges incident to v.

If there are 0 indices, then Lemma 1.3.1 implies that $g_{dG(v)-1}, g_{dG(v)}, g_1$ is also a portion of the clockwise listing in S of edges incident to v. This implies that $g_1, g_2, \ldots, g_{dG(v)}$ is the clockwise listing in S of edges incident to v. In this case $p = 0$, and thus v is not incident to the infinite face of G, so property (ii) of configurations insures that $\gamma_K(v) = d_G(v)$.

Now suppose that v is incident to the infinite face. Property (ii) of configurations implies that $d_G(v) < \gamma_K(v)$ and since K appears in S, $d_S(v) = \gamma_K(v)$. Therefore, there are $p = \gamma_K(v) - d_G(v)$ edges $s_1, \ldots, s_p \in E(S) - E(G)$ incident to v in addition to the edges $g_1, \ldots, g_{d\,G(v)}$. If for $1 \leq i \leq p$, the vertex $v_i \in V(S) - \{v\}$ is an endpoint of si, then $v_i \notin V(G)$, because if it was, then the fact that G is an induced sub drawing of S would imply that $s_i \in E(G)$. We may assume without loss of generality that $s_1, s_2, \ldots, s_p$ is a portion of a clockwise listing in S of edges incident to v. Therefore, since $g_1, g_2, \ldots, g_{dG(v)}$ is a portion of the clockwise listing in S of neighbors incident to v, $g_1, \ldots, g_{dG(v)}, s_1, \ldots, s_p$ is a clockwise listing in S of neighbors incident to v. This completes the proof of the Lemma 1.3.3.

Lemma 1.3.1 *Let K be a configuration that appears in S, where S is either a free completion of K or a triangulation and let $v \in V(G(K))$ be a cut-vertex of $G = G(K)$. Let g1, . . . , $g_{dG(v)}$ be a clockwise listing in G of edges incident to v such that g_1 and $g_{dG(v)}$ are both incident to the infinite face of G. There is a clockwise listing in S of edges incident to v of the following form.*

$g_1, \ldots, g_p, s_1, g_{p+1}, \ldots, g_{dG(v)}, s_2$

where $p < d_G(v)$, $\gamma K(v) = d_G(v) + 2$, and where the endpoints of s_i are in $(V(S) - V(G)) \cap \{v\}$.

Proof: From Lemma 1.3.2 we know that there is an index in $\{1, 2, \ldots, d_G(v) - 1\}$ such that g_p and g_{p+1} are incident to the infinite face of G. Here, we are as usual interpreting g_{i+1} as g_1 when $i = d_G(v)$. Since there are exactly two indices with this property, we know that there exists an integer p such that for $1 \leq l < p$ and $p + 1 \leq l < d_G(v)$ the edges g_l and g_{l+1} are both incident to a common finite face of G. Therefore by Lemma 1.3.1, $g_1, g_2, \ldots, g_p$ and $g_{p+1}, \ldots, g_{dG(v)}$ are portions of the clockwise listing in S of edges incident to v. By property (i) of configurations, and by the fact that K appears in S, $\gamma d_S(v) = \gamma K(v) = d_G(v) + 2$. Let s_1, s_2 be the edges in E(S) - E(G) that are incident to v. If $v_i \in V(S) - \{v\}$ are the endpoints of si for i = 1, 2, then the fact that G is an induced sub drawing of S implies that $v_1, v_2 \in V(S) - V(G)$. Thus there are without loss of generality, three possibilities for the clockwise listing in S of edges incident to v.

1) $g_1, \ldots, g_p, s_1, g_{p+1}, \ldots, g_{dG(v)}, s_2$.
2) $g_1, \ldots, g_p, s_1, s_2, g_{p+1}, \ldots, g_{dG(v)}$.
3) $g_1, \ldots, g_p, g_{p+1}, \ldots, g_{dG(v)}, s_1, s_2. g_{dG(v)}$,

By Lemma 1.2.1, v is not incident to an infinite face of S. Therefore any two consecutive edges in a clockwise listing in S of edges incident to v are incident to a common finite face of S. Thus, if 2) is a clockwise listing in S of edges incident to v, then $g_{dG(v)}$, and g_1 are incident to a common finite triangular face r of S. This means that if $d_G(v)$ and h_1 are the endpoints of $g_{dG(v)}$, and g_1 in $V(G) - \{v\}$, then there is an edge $e \in E(S)$ with endpoints $h_{dG(v)}$ and h_1 and incident to the face r. Now e has both endpoints in G and since G is an induced sub drawing of S, e is an edge of G, and thus r is a face of G. Now it must be that r is designated as the infinite face of G, since by hypothesis, $g_{dG(v)}$, g1 are both incident to the infinite face of G. Therefore, the infinite face of G is a triangular face which is incident to the edges $g_{dG(v)}, g_1$, and e. However, the infinite face of G was also assumed to be incident to the edges g_p and g_{p+1} which are both distinct from the edges $g_{dG(v)}$ and g_1. This is a contradiction. A similar contradiction arises if 3) is a clockwise listing in G. This implies that 1) is a clockwise listing in G of edges incident to v, and this completes the proof of the Lemma.

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