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# 7. A GENERALIZATION OF DIMENSION OF VECTOR SPACES TO MODULES OVER ASSOCIATIVE RINGS

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***(This Survey paper is dedicated in memory of Prof. Dr A.W. Goldie)***

**Abstract**

The aim of the present paper is to provide information regarding the generalization of the concept 'dimension' of vector spaces to modules over associative rings. This paper explains a fundamental structure theorem related to the dimension of modules that was proved in 1957 by a well known algebraist A.W. Goldie (University of Leeds). This paper is very much useful for the beginners in research to understand the algebraic techniques how definitions and other new notions go on, and how logic is generated to form some important structure theorems in algebra.

**Introduction :**

In recent decades interest has arisen in algebraic systems with binary operations addition and multiplication. 'Ring' is one of such systems. A ring (or an associative ring) is an algebraic system $(R, +, .)$ satisfying the conditions:

i) $(R, +)$ is an Abelian group;
ii) $(R, .)$ is a semi-group; and
iii) $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in R$.

Moreover, if there exists an element $1 \in R$ such that $1a = a = a1$ for all $a \in R$, then we say that R is a ring with identity. Ring theory became an important part of Algebra.

We know that the algebraic structure 'vector space' is normally defined over a field. A similar structure defined over an associative ring is called as 'module'. Goldie [1] introduced the concept of Finite Goldie Dimension (FGD, in short) in modules. A module M is said to have FGD if M contains no infinite direct sum of non-zero submodules. Goldie proved a structure theorem for modules which states that "a module with FGD contains uniform submodules $U_1, U_2, \ldots, U_n$ whose sum is direct and essential in M". The number n obtained here is independent of the choice of $U_1, U_2, \ldots, U_n$ and it is called as Goldie Dimension of M. Later this dimension theory was studied/developed by the authors like: Reddy, Satyanarayana, Syam Prasad & Nagaraju. The present paper includes the proof of this structure theorem along with the proofs for some other necessary related results.

This paper is divided into two sections. In section-1, we presented some fundamental definitions and examples on modules which will be useful in the second section. In section-2, we presented some fundamental definitions and results on Finite Goldie Dimension from the existing literature. Finally we presented the proof for the structure theorem related to modules with finite Goldie dimension. For the better understanding for the beginners, we explained the proofs in detail. We do not included the easier verifications.

## SECTION - 1
## Modules

In this section, we give some fundamental definitions and examples on modules which are used in the next section.

**1.1 Definition:** Let $R$ be an associative ring. An Abelian group $(M, +)$ is said to be a ***module*** over $R$ if there exists a mapping (called ***scalar multiplication***) $f : R \times M \to M$ (the image of $(r, m)$ is denoted by $rm$) satisfying the following three conditions:

(i) $r(a+b) = ra + rb$;

(ii) $(r+s)a = ra + sa$; and

(iii) $r(sa) = (rs)a$ for all $a, b \in M$ and $r, s \in R$.

Moreover, if $R$ is a ring with identity $1$; and $1m = m$ for all $m \in M$, then $M$ is called a ***unital R–module*** or ***unitary module***.

**1.2 Note:** In the definition 1.1, we write the elements of $R$ on left to the elements of M. So we call this module as a ***left R-module*** or ***left module***.
Throughout our discussion, M stands for a non-zero left module.

**1.3 Examples:** (i) Every ring $R$ is a module over itself.

(ii) Every Abelian group is a module over the ring of integers $\mathbb{Z}$.

(iii) Every vector space over a field $F$, is a module over the ring $F$.

(iv) Let $(G, +)$ be an Abelian group.

Write $R = \{f : G \to G \ / \ f \text{ is a group homomorphism}\}$.
We define the operations addition (+) and multiplication $(\cdot)$ on $R$ by
$(f + g)(x) = f(x) + g(x)$, and $(f.g)(x) = f(g(x))$ for all $x \in G$ and $f, g \in R$.
Define $0(x) = 0$ for all $x \in G$. Then $0$ is the additive identity in $R$.
For any $f \in R$, we define $-f$ in $R$ by $(-f)(x) = -(f(x))$. Then "$- f$" is the additive inverse of $f$. So $(R, +)$ becomes an additive Abelian group. Also $(R, .)$ is a semi group.
One can easily verify the distributive laws:

$$f(g + h) = fg + fh \text{ and } (f + g)h = fh + gh \text{ for all } f, g, h \in R.$$

The identity function $I(x) = x$ for all $x \in G$, acts as the multiplicative identity.
So $(R, +, .)$ is a ring with the identity I(Here identity function on $G$ acts as the multiplicative identity in $R$). For any $f \in R$ and $a \in G$, the element $f(a)$ (the image of 'a' under f) is in $G$. This defines a scalar multiplication $R \times G \to G$. Now G becomes a module over $R$.

(v) Let $R$ be a ring, and $L$ a left ideal of $R$.
Define $a \sim b \Leftrightarrow a - b \in L$ for any $a, b \in R$. Then $\sim$ is an equivalence relation and the equivalence class containing $a$ is $[a] = a + L$. Write $M = \{a + L / a \in R\}$. Define addition $(a + L) + (b + L) = (a + b) + L$ on M. Then $(M, +)$ is an Abelian group. Here $0 + L$ is the additive identity, and $(- a) + L$ is the inverse of $(a + L)$ in M. For any $r \in R$

and $a + L \in M$, define the scalar multiplication by $r(a + L) = ra + L$. It can easily shown that $M$ is an R-module. Here $M$ is called the ***quotient module*** of $R$ by $L$.

**1.4 Definition:** Let $M$ and $M^1$ be R-modules. A mapping $f : M \to M^1$ is said to be an ***R-module homomorphism*** (or ***module homomorphism***) if f satisfies the following conditions:

(i). $f(x + y) = f(x) + f(y)$; and (ii) $f(rx) = rf(x)$ for all $x,y \in M$ and $r \in R$.

**1.5 Definition:** Let $M$ be an R-Module. A non-empty subset $N$ of $M$ is called an ***R-submodule*** **(**or a ***submodule*****)** of $M$ if

(i) For any $x, y \in N \Rightarrow x - y \in N$; and (ii) $x \in N$ and $r \in R \Rightarrow rx \in N$.

**1.6 Note:** (i). For any module $M$, the sets $\{0\}$ and $M$ are submodules of M. These two submodules are called ***trivial submodules*** of M.
(ii). For a non-empty subset X of M, we define the submodule of M generated by X as the intersection of all submodules of M containing X.
(iii). The submodule of M generated by X is denoted by $<X>$ or $(X)$. We write $<x>$ (instead of $<\{x\}>$ or $(x)$).

**1.7 Definitions:** Let $M$ be an R-module and $M_1, M_2, \ldots, M_s$ submodules of $M$. We say that $M$ is the ***direct sum*** of $M_i$, $1 \le i \le s$ if every element $m \in M$ can be expressed in unique manner as $m = m_1 + m_2 + \ldots + m_s$ with $m_i \in M_i$, $1 \le i \le s$.
A Module $M$ is said to be ***cyclic*** if there exists an element $a \in M$ such that $M = \{ra \ / \ r \in R\}$. An R-module $M$ is said to be ***finitely generated*** if there exist elements $a_j \in M$, $1 \le j \le n$ such that $M = \{r_1a_1 + \ldots + r_na_n \ / \ r_j \in R$, for $1 \le j \le n\}$.
It is clear that every cyclic module is finitely generated.

**1.8 Definition**: (i) If $K$, $A$ are submodules of $M$, and $K$ is a maximal submodule of $M$ with respect to the property $K \cap A = (0)$, then $K$ is said to be a ***complement*** of $A$ (or a ***complement submodule*** in M).

(ii) A non-zero submodule $K$ of $M$ is called ***essential*** (or ***large***) in $M$ (or M is an ***essential extension*** of K) if $A$ is a submodule of $M$ such that $K \cap A = (0)$, imply $A = (0)$.

(iii) If $A$ is essential submodule of $M$ and $A \neq M$, then we say that $M$ is a ***proper essential extension*** of A.

**1.9 Note**: (i) If $K$ is essential in M, then we denote this fact by $K \le_e M$.
(ii) Let $M$ be a module, $A$ a submodule of $M$ and $B$ be a maximal among the submodules of $M$ with respect to the property that $A \cap B = (0)$. Then $A \oplus B \le_e M$, and $B$ is a complement of A.

[**Verification**: Suppose D is a submodule of M such that $(A + B) \cap D = (0)$.
To show $D = (0)$, first we show that $A \cap (B + D) = (0)$.
Let $a \in A \cap (B + D) \Rightarrow a = b + d$ for some $b \in B$ and $d \in D$.
$\Rightarrow a - b = d \in D \cap (A + B) = (0) \Rightarrow a - b = 0 \Rightarrow a = b \in A \cap B = (0)$.
Since $A \cap (B + D) = (0)$, by maximality of B, we have
$B = B + D \Rightarrow D \subseteq B \subseteq A + B \Rightarrow D \subseteq (A + B) \cap D = (0)$. Hence $D = (0)$.
This shows that $A \oplus B \le_e M$.]

(iii) If $C$ is a submodule of $M$ which is maximal with respect to the property: $C \supseteq A$ and $C \cap B = (0)$, then $C \oplus B \leq_e M$. Moreover $C$ is a complement of $B$ containing $A$. [**Verification**: Follows from (ii)].

**1.10 Result:** (i). The intersection of finite number of essential submodules is essential;

(ii). If $I$, $J$, $K$ are submodules of $M$ such that $I \leq_e J$, and $J \leq_e K$, then $I \leq_e K$;

(iii). $I \leq_e J \Rightarrow (I \cap K) \leq_e (J \cap K)$;

(iv). If $I \subseteq J \subseteq K$, then $I \leq_e K \Leftrightarrow I \leq_e J$, and $J \leq_e K$; and

(v). Suppose $H$, $K$ are two modules and $f : K \to H$ is a module isomorphism. If $A$ is a submodule of $K$, then $A \leq_e K \Leftrightarrow f(A) \leq_e H$.

**Proof:** (i) Let $\{A_i\}_{i=1}^{n}$ be a family of essential submodules of $M$.

Put $A = \bigcap_{i=1}^{n} A_i$. We show that $A \leq_e M$.

We use the principle of mathematical induction on $n$, the number of essential submodules. Suppose $n = 2$. Then $A = A_1 \cap A_2$. Let $L$ be a submodule of $M$ such that $A \cap L = (0) \Rightarrow (A_1 \cap A_2) \cap L = (0) \Rightarrow A_1 \cap (A_2 \cap L) = (0)$

$\Rightarrow A_2 \cap L = (0)$ (since $A_1 \leq_e M$) $\Rightarrow L = (0)$ (since $A_2 \leq_e M$).

Now we proved that $A \cap L = (0)$, $L$ is a submodule of $M \Rightarrow L = (0)$.

Hence $A = (A_1 \cap A_2) \leq_e M$. So the result is true for $n = 2$.

**Induction Hypothesis*:*** Assume that the result is true for $(n - 1)$ submodules.

That is, $\bigcap_{i=1}^{n-1} A_i \leq_e M$ whenever $A_i \leq_e M$, $1 \leq i \leq n - 1$.

Now suppose that $A_i$, $1 \leq i \leq n$ are essential submodules of $M$. We have to prove that $\bigcap_{i=1}^{n} A_i \leq_e M$. Let $L$ be a submodule of $M$ such that $\left(\bigcap_{i=1}^{n} A_i\right) \cap L = (0)$

$$\Rightarrow \left(\bigcap_{i=1}^{n-1} A_i \cap A_n\right) \cap L = (0) \Rightarrow \left(\bigcap_{i=1}^{n-1} A_i\right) \cap (A_n \cap L) = (0)$$

$\Rightarrow A_n \cap L = (0)$ [by induction hypothesis] $\Rightarrow L = (0)$ [since $A_n \leq_e M$].

Hence $\bigcap_{i=1}^{n} A_i \leq_e M$. This proves that the result is true for all positive integers $n$.

(ii). Suppose $I$, $J$, $K$ are submodules of $M$ such that $I \leq_e J$ and $J \leq_e K$.

Now we show that $I \leq_e K$.

Let $L$ be a submodule of $M$ such that $L \subseteq K$ and $I \cap L = (0)$

$\Rightarrow (I \cap L) \cap J = (0) \Rightarrow I \cap (L \cap J) = (0)$

$\Rightarrow L \cap J = (0)$ (since $L \cap J \subseteq J$ and $I \leq_e J$) $\Rightarrow L = (0)$ (since $J \leq_e K$).

This shows that $I \leq_e K$.

(iii) Suppose $I \leq_e J$. Now we show that $(I \cap K) \leq_e (J \cap K)$.

For this, take $L$ a submodule of $M$ such that $L \subseteq J \cap K$ and $(I \cap K) \cap L = (0)$

$\Rightarrow I \cap (K \cap L) = (0) \Rightarrow K \cap L = (0)$ (since $K \cap L \subseteq L \subseteq J \cap K \subseteq J$ and $I \leq_e J$)

$\Rightarrow L = K \cap L = (0)$ [since $L \subseteq K$] $\Rightarrow L = (0)$. Therefore $(I \cap K) \leq_e (J \cap K)$.

(iv) Suppose $I \subseteq J \subseteq K$ and $I \leq_e K$.

Let $L$ be a submodule of $M$ such that $L \subseteq J$ and $I \cap L = (0)$,
$\Rightarrow I \cap L = (0)$, $L \subseteq K$ [since $L \subseteq J \subseteq K$] $\Rightarrow L = (0)$ [since $I \leq_e K$].
This shows that $I \leq_e J$. Next we show that $J \leq_e K$.
For this, take a submodule $A$ of $M$ such that $A \subseteq K$ and $J \cap A = (0)$.
Now $I \subseteq J \Rightarrow I \cap A \subseteq J \cap A = (0) \Rightarrow I \cap A = (0) \Rightarrow A = (0)$ [since $A \subseteq K$ and $I \leq_e K$]. We proved that $J \cap A = (0)$, $A$ is a submodule of $K \Rightarrow A = (0)$.
Hence $J \leq_e K$. The converse part follows from (ii).

(v). Suppose $A \leq_e K$. We have to show that $f(A) \leq_e H$. Let $B$ be a submodule of $H$ such that $f(A) \cap B = (0)$. Now $A \cap f^{-1}(B) = (0)$. Since $A \leq_e K$, we have $f^{-1}(B) = (0)$ and so $B = f(f^{-1}(B)) = (0)$. Thus $f(A) \leq_e H$.
If $f(A) \leq_e H$. By above part $f^{-1}(f(A)) \leq_e f^{-1}(H) = K \Rightarrow A \leq_e K$.

**1.11 Lemma**: Let $G_1, G_2, \ldots G_n, H_1, H_2, \ldots H_n$ be submodules of $M$ such that the sum $G_1 + G_2 + \ldots + G_n$ is direct and $H_i \subseteq G_i$ for $1 \leq i \leq n$. Then
$(H_1 + H_2 + \ldots + H_n) \leq_e (G_1 + G_2 + \ldots + G_n) \Leftrightarrow$ each $H_i \leq_e G_i$ for $1 \leq i \leq n$.

**Proof**: We prove the Lemma by induction on $n$.
Suppose that $n = 2$. Assume that $H_1 \leq_e G_1$ and $H_2 \leq_e G_2$. Write $A_1 = H_1 + G_2$ and $A_2 = G_1 + H_2$. We show that $A_1 \leq_e (G_1 + G_2)$.
For this, it is enough to show that $A_1 \cap Ra \neq (0)$ for all $0 \neq a \in G_1 + G_2$.
Let $0 \neq a \in G_1 + G_2$. Then $a = a_1 + a_2$ for some $a_1 \in G_1$ and $a_2 \in G_2$.
If $a_1 = 0$, then $a \in G_2$ and hence $\langle a\rangle \cap A_1 \neq (0)$.
If $a_1 \neq 0$, then since $H_1 \leq_e G_1$, we have $Ra_1 \cap H_1 \neq (0)$.
So there exists a non-zero element $r \in R$ such that $0 \neq ra_1 \in Ra_1 \cap H_1$.
If $r(a_1 + a_2) = 0$, then $ra_1 = -ra_2 \in G_1 \cap G_2 = (0) \Rightarrow ra_1 = 0$, a contradiction.
Now $0 \neq r(a_1 + a_2) = ra_1 + ra_2 \in H_1 + G_2 = A_1$.
So $0 \neq ra = r(a_1 + a_2) \in A_1 \cap Ra$. Hence $A_1 \leq_e (G_1 + G_2)$.
Similarly $A_2 \leq_e (G_1 + G_2)$. By Result 1.10(i), it follows that
$(H_1 + H_2) = (A_1 \cap A_2) \leq_e (G_1 + G_2)$. So the result is true for two submodules.
**Induction Hypothesis**: Assume the result for $k - 1$ submodules.
That is, if $H_i \leq_e G_i$ $1 \leq i \leq k - 1$, then $(H_1 + \ldots + H_{k-1}) \leq_e (G_1 + G_2 + \ldots + G_{k-1})$.
Now $H_k \leq_e G_k$. By applying above part we get that

$$(H_1 + H_2 + \ldots + H_{k-1} + H_k) \leq_e (G_1 + G_2 + \ldots + G_{k-1} + G_k).$$

Hence the result is true for $k$ submodules.
Hence by the principle of mathematical induction, for any positive integer $n$,

$$(H_1 + H_2 + \ldots + H_n) \leq_e (G_1 + G_2 + \ldots + G_n) \text{ if } H_i \leq_e G_i,\ 1 \leq i \leq n.$$

**Converse**: The converse also proved by using mathematical induction.
Let $G_1, G_2, H_1, H_2$ be submodules of an R-module $M$ such that $G_1 \cap G_2 = (0)$ and $H_i \subseteq G_i$ for $i = 1, 2$. If $(H_1 + H_2) \leq_e (G_1 + G_2)$, then $H_i \leq_e G_i$ for $i = 1, 2$.
**[**Otherwise: Suppose $(H_1 + H_2) \leq_e (G_1 + G_2)$ and $H_1$ is not essential in $G_1$.
Then there exists a non-zero submodule $A$ of $M$ such that $A \subseteq G_1$ and $A \cap H_1 = (0)$.
To show $A \cap (H_1 + H_2) = (0)$. Let $x \in A \cap (H_1 + H_2)$. Then $x \in A$ and $x \in H_1 + H_2$.
$x = h_1 + h_2$ for some $h_1 \in H_1$ and $h_2 \in H_2$.
Now $-h_1 + x = h_2 \in (H_1 + A) \cap H_2 \subseteq G_1 \cap G_2 = (0)$ and hence $h_2 = 0$.
$\Rightarrow x = h_1 \in H_1 \cap A = (0)$ so that $x = 0$.
Therefore $A \cap (H_1 + H_2) = (0)$. Hence $(H_1 + H_2) \leq_e (G_1 + G_2)$ and $A \subseteq G_1 + G_2$ we have $A = (0)$, a contradiction. Thus $H_1 \leq_e G_1$. In a similar way, we can show that $H_2 \leq_e G_2$**]**.
Thus we can be easily proceed to show that for any position integer $n$,
$(H_1 + H_2 + \ldots + H_n) \leq_e (G_1 + G_2 + \ldots + G_n)$ imply $H_i \leq_e G_i$, for $1 \leq i \leq n$.

**1.12 Note:** Consider submodules A, B, C of M as in Note 1.9(ii) & (iii). Here $A\oplus B \leq_e M$ and $A\oplus B \subseteq C\oplus B \subseteq M$. Using Result 1.10(iv), we get that $A\oplus B \leq_e C\oplus B$. By Lemma 1.11, it follows that $A \leq_e C$. Note that C is a complement submodule which is also an essential extension of A.

## SECTION - 2
## Finite Goldie Dimension In Modules

It is well known that the dimension of a vector space is defined as the number of elements in its basis. One can define a basis of a vector space as a maximal set of linearly independent vectors or a minimal set of vectors which spans the space. The former when generalized to modules over rings, becomes the concept of "Goldie Dimension".

In this section, we present some fundamental definitions and results on Finite Goldie Dimension from the existing literature. We include the proofs for some results. We start this section with the following definition:
R denotes a fixed (not necessarily commutative) ring.

**2.1 Definition**: A module M is said to have ***Finite Goldie Dimension*** (written as **FGD**) if M does not contain a direct sum of infinite number of non-zero submodules.

**2.2 Remark**: The following two conditions are equivalent:

(i) M has FGD; and

(ii) For any strictly increasing sequence $H_0 \subseteq H_1 \subseteq \ldots$ of submodules of M, there exists an integer i such that $H_k \leq_e H_{k+1}$ for every $k \geq i$.

**Verification: (i) ⇒ (ii):** Let M be a module with FGD and $H_0 \subseteq H_1 \subseteq \ldots$ a strictly increasing sequence of submodules of M. In a contrary way, suppose that there exists an infinite sub sequence $\{H_{k_i}\}_{i=1}^{\infty}$ of $\{H_i\}_{i=1}^{\infty}$ such that $H_{k_i}$ is not essential in $H_{k_{i+1}}$, for each i. So for each i, there exists a non-zero submodule $A_i$ of M such that $A_i \subseteq H_{k_{i+1}}$ and $H_{k_i} \cap A_i = (0)$.

Now it is easy to verify that the sum $\sum_{i=1}^{\infty} A_i$ is direct sum of non-zero submodules $\{A_i / i = 1, 2, ...\}$ of M, a contradiction to the fact that M has FGD.

**(ii) ⇒ (i)**: Suppose (ii). In a contrary way, suppose that "M has no FGD".

Then M contains a direct sum of an infinite number of non-zero submodules $\{A_i\}_{i=1}^{\infty}$.

That is, $M \supseteq A_1 \oplus A_2 \oplus A_3 \oplus \ldots$ Now it is clear that $A_1 \subsetneq A_1 \oplus A_2 \subsetneq \ldots$
Since $A_2 \neq (0)$, we have that $A_1$ is not essential in $A_1 \oplus A_2$.
Similarly $A_1\oplus \ldots \oplus A_k$ is not essential in $A_1\oplus A_2\oplus \ldots \oplus A_k\oplus A_{k+1}$.

Write $H_k = A_1\oplus \ldots \oplus A_k$ for $k = 1, 2, \ldots$ Now $H_1 \subsetneq H_2 \subsetneq \ldots$ is an infinite chain of submodules of M such that $H_k$ is not essential in $H_{k+1}$, a contradiction to the converse hypothesis. Hence M has FGD.

**2.3 Note**: (i) If M has DCC (Descending Chain Condition) on its submodules, then M has FGD.
[**Verification**: Suppose M has DCC on its submodules. In a contrary way, suppose M does not have FGD. Then M contain non-zero submodules $K_1, K_2, \ldots$ whose sum is direct. The sequence

$\sum_{i=1}^{\infty} K_i \supseteq \sum_{i=2}^{\infty} K_i \supseteq \ldots\ldots\ldots$ is an infinite strictly decreasing sequence of submodules of M, a contradiction to the fact that M has DCC on its submodules. This proves that M has FGD.]

(ii). If M has ACC (Ascending Chain Condition) on its submodules, then M has FGD.
[**Verification**: Suppose M has ACC on its submodules. Then M satisfies condition (ii) of Remark 2.2. Now one can conclude that M has FGD.]

(iii) Let M be a module with FGD and H be a non-zero submodule of M. Then H has FGD.
[**Verification**: In a contrary way, suppose that H do not have FGD. Then H contains a direct sum of infinite number of non-zero submodules. Since every submodule of H is also a submodule of M, we have M contains an infinite number of non-zero submodules whose sum is direct, a contradiction to the fact that M has FGD. Therefore H has FGD.]

**2.4 Definition**: A non-zero submodule K of M is said to be an ***uniform submodule*** if every non-zero submodule of K is essential in K.

**2.5 Theorem**: (i) K is an uniform submodule $\Leftrightarrow$ $K_1$ , $K_2$ are two submodules of K such that $K_1 \cap K_2 = (0)$ $\Rightarrow$ $K_1 = (0)$ or $K_2 = (0)$.

(ii) Let K and H be two submodules and $f : K \to H$ be module isomorphism. If U is submodule of K, then U is uniform in K $\Leftrightarrow$ f(U) is uniform in H.

(iii) Let H and K be two submodules of M such that $H \cap K = (0)$. For a submodule U of H, we have that U is uniform $\Leftrightarrow$ $(U + K)/K$ is uniform in M/K.

**Proof:** (i) Suppose K is an uniform submodule of M.
Let $K_1$, $K_2$ are two submodules of K such that $K_1 \cap K_2 = (0)$. If $K_1 = (0)$, then it is clear. Suppose $K_1 \neq (0)$. Then since $K_1$ is a submodule of K, we have that $K_1 \leq_e K$ and so $K_2 = (0)$. Thus either $K_1 = (0)$ or $K_2 = (0)$.
**Converse**: Let $K_1$ be a non-zero submodule of K. To verify that $K_1 \leq_e K$, take a submodule $K_2$ of K such that $K_1 \cap K_2 = (0)$. Now by converse hypothesis, we get that $K_2 = (0)$. This shows that $K_1 \leq_e K$. Hence K is uniform.

(ii) Suppose that U is uniform in K. Let I and J be two submodules of H contained in f(U) such that $I \cap J = (0)$. Since f is an isomorphism, $f^{-1}(I)$ and $f^{-1}(J)$ are submodules of K contained in U. $f^{-1}(I) \cap f^{-1}(J) = f^{-1}(I \cap J) = f^{-1}(0) = 0$.
Since U is uniform, $f^{-1}(I) = (0)$ or $f^{-1}(J) = (0)$ $\Rightarrow$ $I = (0)$ or $J = (0)$.

**Converse**: Suppose f(U) is uniform. Since $f^{-1} : H \to K$ is an isomorphism, we get that $f^{-1}(f(U))$ is uniform in K. Since $U = f^{-1}(f(U))$, we have that U is uniform in K.

(iii) Define $f : H \to (H + K)/K$ by $f(h) = h + K$. Clearly f is onto.
Since $H \cap K = (0)$, we have that f is one-one. So f is a module isomorphism.
By (ii) we get that U is uniform in H $\Leftrightarrow$ $f(U) = (U + K)/K$ is uniform in $(H + K)/K$.

**2.6 Lemma**: Let M be a non-zero module with FGD. Then every non-zero submodule of M contains a uniform submodule.

**Proof:** In a contrary way, suppose that U is a non-zero submodule of M which contains no uniform submodule of M. Then U is not uniform.
Therefore there exist two non-zero submodules $U_1$, $U_1^1$ of M such that $U_1 \cap U_1^1 = (0)$, and $U_1 + U_1^1 \subseteq U$.
By our supposition $U_1^1$ is not uniform and this implies that there exist two non-zero submodules $U_2$, $U_2^1$ of M such that $U_2 \cap U_2^1 = (0)$, and $U_2 + U_2^1 \subseteq U$.

If we continue this process, we get two infinite sequences $\{U_i\}_{i=1}^{\infty}$ , $\{Ui^1\}_{i=1}^{\infty}$ of nonzero submodules of M such that $U_i \cap U_i^1 = (0)$ for each i and $U_i + U_i^1 \subseteq U_{i-1}^1$ for $i \geq 2$

Now the sum $\sum_{i=1}^{\infty} U_i$ is an infinite direct sum of non-zero submodules of M,

a contradiction to the fact that M has FGD. Thus we proved that if M has FGD, then every non-zero submodule of M contains a uniform submodule.

**2.7 Remark**: Let M be a module and K a uniform submodule of M.
If L is a submodule of M such that $L \subseteq K$, then either $L = (0)$ or L is uniform.

[**Verification**: Suppose $L \neq (0)$. Let A and B be submodules of M such that $A \subseteq L$, $B \subseteq L$ and $A \cap B = (0)$. Now $A, B \subseteq K$ (since $L \subseteq K$). Since K is uniform, we have that $A = (0)$, or $B = (0)$. This show that L is uniform.]
With the concepts defined above, Goldie proved the following Theorem.

**2.8 Theorem**: (**Goldie [ 1 ]**): Let M be a module with finite Goldie dimension.

(i) (***Existence***) There exist uniform submodules $U_1, U_2, \ldots, U_n$ whose sum is direct and essential in M.

(ii) If J is a submodule of M such that $J \cap U_i \neq (0)$ for all i $(1 \leq i \leq n)$, then J is essential in M.

(iii) (***Uniqueness***) If there exist uniform submodules $V_1, V_2, \ldots, V_k$ whose sum is direct and essential in M, then $k = n$.

**2.9 Definition:** Let M be a module with FGD. Then by Theorem 2.8, there exist uniform submodules $U_i$, $1 \leq i \leq n$ whose sum is direct and essential in M. The number 'n' is independent of the choice of the uniform submodules. This number n is called the ***Goldie dimension*** of M, and it is denoted by ***dim M***.

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