

5. RAINBOW RAMSEY NUMBERS

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Abstract:

This paper studies the Rainbow Ramsey Number for a non empty graph and the main results are 1. The Rainbow Ramsey Number of a graph F with out isolated vertices is defined if and only if F is a forest. 2. The Rainbow Ramsey Number of two graphs F_1 and F_2 with out isolated vertices is defined if and only if F_1 is a star or F_2 is a forest.

Mathematics Subject Classification 2000: 05CXX, 05C55, 05DXX, 05D10, 04XX, 04A10

Key words: Rainbow Ramsey Number, forest, isolated vertices, star

1. Introduction:

Basically in an edge-colored graph G that if there is a sub graph F of G all of whose edges are colored the same, then F is referred to as a monochromatic F . On the other hand, if all edges of F are colored differently, then F is referred to as a rainbow F . Arie Bialostocki and William voxman [1] defined, the **Rainbow Ramsey Number** $RR(F)$ for a non empty graph F .

2. Definition: For a nonempty graph F , the **Rainbow Ramsey Number** $RR(F)$ of F as the smallest positive integer n such that if each edge of the complete graph K_n is colored from any set of colors, then either a monochromatic F or a rainbow F is produced.

Let $\{v_1, v_2 \dots V_n\}$ be the vertex set of a complete graph K_n . An edge coloring of K_n using positive integers for colors is called a **minimum coloring** if two edges $v_i v_j$ and $v_k v_l$ are colored the same if and only if $\text{Min} \{i, j\} = \{k, l\}$

while an edge coloring of K_n is called a **maximum coloring** if two edges $u_i u_j$ and $u_k u_l$ are colored the same if and only if $\text{Max} \{i, j\} = \text{max} \{k, l\}$

In 1950 Paul Erdos and Richard Rado [3] showed that if the edges of a sufficiently large complete graph are colored from a set of positive integers, then there must be a complete sub graph of prescribed order that is monochromatic or rainbow or has a minimum or maximum coloring.

Note: For a nonempty graph F , the **Rainbow Ramsey Number** $RR(F)$ of F as the smallest positive integer n such that if each edge of the complete graph K_n is colored from any set of positive integers, then there is a complete sub graph of order p that is either monochromatic or rainbow or has a minimum or maximum coloring.

With the aid of above note Arie Bialostocki and William voxman [1] determined all those graphs F for which $RR(F)$ is defined.

2.1 Definition: A graph with out cycles is a forest

2.2 Theorem: Let F be a graph without isolated vertices. The Rainbow Ramsey number $RR(F)$ is defined if and only if F is a forest.

Let F be a graph of order $p \geq 2$. First we show that $RR(F)$ is defined only if F is a forest. Suppose that F is not a forest. Thus F contains a cycle C , of length $k \geq 3$ say. Let n be an integer with $n \geq p$ and let $\{v_1, v_2, \dots, v_n\}$ be the vertex set of a complete graph K_n . Define an edge coloring c of K_n by $c(v_i v_j) = i$ if $i < j$. Hence c is a minimum edge coloring of K_n . If k is the minimum positive integer such that v_k belongs to C , then two edges of C are colored k , implying that there is no rainbow F in K_n . Since any other edge in C is not colored k , it follows that F is not monochromatic either. Thus $RR(F)$ is not defined.

For the converse, suppose that F is a forest of order $p \geq 2$. By known fact there exists and integer $n \geq p$ such that for any edge coloring of K_n with positive integers, there is a complete subgraph G of order p in K_n that is either monochromatic or rainbow or has minimum or maximum coloring. If G is monochromatic or rainbow, then K_n contains a monochromatic or rainbow F . Hence we may assume that the edge coloring of G is minimum or maximum, say the former. We show in this case that G contains a rainbow F . If F is not a tree, then we can add edges to F to produce a tree T of order p .

Let $V(G) = \{v_{i_1}, v_{i_2}, \dots, v_{i_p}\}$,

Where $i_1 < i_2 < \dots < i_p$. Select some vertex $v = v_{i_p}$ of T and label the vertices of T in the order $v = v_{i_p}, v_{i_{p-1}}, \dots, v_{i_2}, v_{i_1}$

of non decreasing distance from v ; that is, $d(v_{ij}, v) \geq d(v_{ij+1}, v)$

for every integer j with $1 \leq j \leq p - 1$. Hence there exists exactly on edge of T having color i_j for each j with $1 \leq j \leq p - 1$. Thus T and hence F is rainbow. The rainbow Ramsey number $RR(F)$ is therefore defined. The following example is due to S.Fiorini, edge colorings of graphs[5].

2.3 Example: For each integer $k \geq 2$, $RR(K_{1,k}) = (k - 1)^2 + 2$.

Proof

. We first show that $RR(K_{1,k}) \geq (k - 1)^2 + 2$. Let $n = (k - 1)^2 + 1$.

We consider two cases.

Case 1. k is odd. Then n is odd Factor K_n into $n-1/2 = (k-1)^2/2$ Hamiltonian cycles each. Partition these cycles into $k - 1$ sets S_i ($1 \leq i \leq k - 1$) of $k-1/2$ Hamiltonian cycles each. Color each edge of each cycle in S_i with color i . then there is neither a monochromatic $K_{1,k}$ nor a rainbow $K_{1,k}$.

Case 2. k is even. Then n is even. Factor K_n into $n-1 = (k-1)^2 - 1$ 1-factors. Partition these 1-factors into $k - 1$ sets S_i ($1 \leq i \leq k - 1$) of $k-1$ 1-factors. Color each edge of each 1-factor in S_i color with i . Then there is neither a monochromatic $K_{1,k}$ nor a rainbow $K_{1,k}$.

Therefore, $RR(K_{1,k}) \geq (k-1)^2 + 2$. It remains to show that $RR(K_{1,k}) \leq (k - 1)^2 + 2$. Let $N = (k - 1)^2 + 2$ and let there be given an edge coloring of K_N from any set of colors. Suppose that no monochromatic $K_{1,k}$ results. Let v be a vertex of K_N . Since $\deg v = N - 1$ and there is no monochromatic $K_{1,k}$, at most $k - 1$ edges incident with v can be colored the same. Thus there are

at least $\lceil \frac{N}{k-1} \rceil = k$ edges incident with v that are colored differently, producing a rainbow $K_{1,k}$.

in view of theorem 2.2, it wouldn't expected that $RR(F_1, F_2)$ is defined for every pair F_1, F_2 of non empty graphs. This is, in fact, the case. While it is a consequence of a result of Erdos and Rado [3] for which graphs F_1 and F_2 the Rainbow Ramsey number $RR(F_1, F_2)$ exists, the proof of the following is due to Linda Eroh [4].

3 Definition: If the partite sets u & w of a complete bi partite graph contain s & t vertices. Then this graph is denoted by $K_{s,t}$, the graph $K_{1,t}$ is called star.

3.1 Theorem: Let F_1 and F_2 be two graphs without isolated vertices. The rainbow Ramsey number $RR(F_1, F_2)$ is defined if and only if F_1 is a star or F_2 is a forest.

Proof. First, we show that $RR(F_1, F_2)$ is exists only if F_1 is a star or F_2 is a forest. Suppose that F_1 is not a star and F_2 is not a forest. Let G be a complete graph of some order n such that $V(G) = \{v_1, v_2, \dots, v_n\}$ and such that both F_1 and F_2 are sub graphs of G . Define an $(n-1)$ -edge coloring on G such that the edge $v_i v_j$ is assigned the color i if $i < j$. Hence this coloring is a minimum edge coloring of G .

. Let G_1 be any copy of F_1 in G and let a be the minimum integer such that v_a is a vertex of G_1 . Then every edge incident with v_a is colored a . since G_1 is not a star, some edge of G_1 is not incident with v_a and is therefore not colored a . Hence G_1 is not monochromatic. Next, let G_2 be any copy of F_2 in G . Since G_2 is not a forest, G_2 contains a cycle C . Let b be the minimum integer such that v_b is a vertex of G_2 belonging to C . Since the two edges of C incident with v_b are colored b (and G_2 contains at least two edges colored b), G_2 is not a rainbow subgraph of G . Hence $RR(F_1, F_2)$ is not defined.

We now verify the converse. Let F_1 and F_2 be two graphs without isolated vertices such that either F_1 is a star or F_2 is a forest. We show that there exists a positive integer n such that for every edge coloring of K_n , either a monochromatic F_1 or a rainbow F_2 results. Suppose that the order of F_1 is $s+1$ and the order of F_2 is $t+1$ for positive integers s and t . Hence $F_1 = K_{1,s}$, according to G.Chartrand, rainbow connection in graphs[2], We now consider two cases , depending on whether F_1 is a star or F_2 is a forest. It is convenient to begin with the case where F_2 is a forest.

Case 1. F_2 is a forest. If F_2 is not a tree, then we may add edges to F_2 so that a tree G_2 results. If F_2 is a tree, then let $G_2 = F_2$. Furthermore, if F_1 is not complete, then we may add edges to F_1 so that a complete graph $G_1 = K_{s+1}$ results. If F_1 is complete, then let $G_1 = F_1$. Hence $G_1 = K_{s+1}$ and G_2 is a tree of order $t+1$. We now show that $RR(G_1, G_2)$ is defined by establishing the existence of a positive integer n such that any edge coloring of K_n from any set of colors results in either a monochromatic G_1 or a rainbow G_2 . This, in turn, implies the existence of monochromatic F_1 or a rainbow F_2 . We now consider two sub cases, depending on whether G_2 is a star.

Sub case 1.1. G_2 is a star of order $t+1$, that is, $G_2 = K_{1,t}$. Therefore, in this subcase, $G_1 = K_{s+1}$ and $G_2 = K_{1,t}$. (This subcase will aid us later in the project) In this subcase, let

$$n = \sum_{i=0}^{(s-1)(t-1)+1} (t-1)^i$$

and let an edge coloring of K_n be given from any set of colors. If K_n contains a vertex incident with t or more edges assigned distinct colors, then K_n contains a rainbow G_2 . Hence we may assume that every vertex of K_n is incident with at most $t-1$ edges assigned distinct colors. Let v_1 be a vertex of K_n . Since the degree of v_1 in K_n is $n-1$, there are at least

$$\frac{n-1}{t-1} = \sum_{i=0}^{(s-1)(t-1)} (t-1)^i$$

edges incident with u_1 that are assigned the same color, say color c_1 .

Let S_1 be the set of vertices joined to v_1 by edges colored c_1 and let $v_2 \in S_1$.

There are at least

$$\frac{|S_1|-1}{t-1} \geq \sum_{i=0}^{(s-1)(t-1)-1} (t-1)^i$$

edges of the same color, say color c_2 , joining v_2 and vertices of S_1 , where possibly $c_2 = c_1$. Let S_2 be the set of vertices in S_1 joined to v_2 by edges colored c_2 . Continuing in this manner, we construct sets $S_1, S_2, \dots, S_{(s-1)(t-1)}$ and vertices $v_1, v_2, \dots, v_{(s-1)(t-1)+1}$ such that $2 \leq i \leq (s-1)(t-1)+1$, the vertex v_i belongs to S_{i-1} and is joined to at least

$$\frac{|S_{i-1}|-1}{t-1} \geq \sum_{i=0}^{(s-1)(t-1)-1} (t-1)^i$$

vertices of S_{i-1} by edges colored c_i . Finally, in the set $S_{(s-1)(t-1)}$, the vertex $v_{(s-1)(t-1)+1}$ is joined to a vertex $v_{(s-1)(t-1)+2}$ in $S_{(s-1)(t-1)}$ by an edge colored $c_{(s-1)(t-1)+1}$. Thus we have a sequence

$$v_1, v_2, \dots, v_{(s-1)(t-1)+2}$$

of vertices such that every edge $v_i v_j$ for $1 \leq i < j \leq (s-1)(t-1)+2$ is colored c_i and where the colors $c_1, c_2, \dots, c_{(s-1)(t-1)+1}$ are not necessarily distinct. In the complete subgraph H of order $(s-1)(t-1)+2$ induced by the vertices listed in (11.3), the vertex $v_{(s-1)(t-1)+2}$ is incident with at most $t-1$ edges having distinct colors. Hence there is a set of at least

$$\left\lceil \frac{(s-1)(t-1)+1}{t-1} \right\rceil = s$$

Vertices in H joined to $v_{(s-1)(t-1)+2}$ by edges of the same color. Let $v_{i1}, v_{i2}, \dots, v_{is}$ be s of these vertices, where $i_1 < i_2 < \dots < i_s$. Then $c_{i1} = c_{i2} = \dots = c_{is}$, and the complete subgraph of order $s+1$ induced by $\{v_{i1}, v_{i2}, \dots, v_{is}, v_{(s-1)(t-1)+2}\}$ is monochromatic.

Subcase 1.2 G_2 is a tree of order $t+1$ that is not necessarily a star. Recall that $G_1 = K_{s+1}$. We proceed by induction on the positive integer t . If $t=1$ or $t=2$, then G_2 is a star and the base case of the induction follows by subcase 1.1. Suppose that $RR(G_1, G_2)$ exists for $G_1 = K_{s+1}$ and for every tree G_2 of order $t+1$ where $t \geq 2$. Let T be a tree of order $t+2$. We show that $RR(G_1, T)$ exists. Let v be an end-vertex of T and let u be the vertex of T that is adjacent to v . Let $T^1 = T - v$. Since T^1 is a tree of order $t+1$, it follows by the induction hypothesis that $RR(G_1, T^1)$ exists, say $RR(G_1, T^1) = p$. Hence for any edge coloring of K_p from any set of colors, there is either a monochromatic $G_1 = K_{s+1}$ or a rainbow T^1 . From sub case 1.1, we know that $RR(G_1, K_{1,t+1})$ exists. Suppose that $RR(G_1, K_{1,t+1}) = q$ and let $n = pq$ in this subcase.

Let there be given an edge coloring of K_n using any number of colors. Consider a partition of the vertex set of K_n into q mutually disjoint sets of p vertices each. By the induction hypothesis, the complete subgraph induced by each set of p vertices contains either a monochromatic K_{s+1} or rainbow T^1 . If a monochromatic K_{s+1} occurs in any of these complete subgraph K_p , then subcase 1.2 is verified. Hence we may assume that there are q pair wise mutually rainbow copies.

$$T_1^1, T_2^1 \dots T_q^1$$

of T^1 , where u_i is the vertex in T_i^1 ($1 \leq i \leq q$) corresponding to the vertex u in T^1 . Let H be the complete subgraph of order q induced by $\{u_1, u_2, \dots, u_q\}$. Since $RR(K_{s+1}, K_{1,t+1}) = q$, it follows that either H contains a monochromatic K_{s+1} or a rainbow $K_{1,t+1}$. If H contains a monochromatic K_{s+1} , then once again, the proof of subcase 1.2 is complete. So we may assume that H contains a rainbow $K_{1,t+1}$. Let u_j be the center of a rainbow star $K_{1,t+1}$ in H . At least one of the $t+1$ colors of the edges of $K_{1,t+1}$ is different from the colors of the t edges of T_j^1 . Adding the edge having this color at u_j in T_j^1 produces a rainbow copy of T .

Case 2. F_1 is a star. Denote F_1 by G_1 as well and so $G_1 = K_{1,s}$. If F_2 is complete, then let $G_2 = F_2$. If F_2 is not complete, then we may add edges to F_2 so that a complete graph $G_2 = K_{t+1}$ results. We verify that $RR(G_1, G_2)$ exists by establishing the existence of a positive integer n such that for any edge coloring of K_n from any set of colors, either a monochromatic G_1 or a rainbow G_2 results. This then shows that K_n will have a monochromatic F_1 or a rainbow F_2 . For positive integers p and r with $r < p$, let

$$p^{(r)} = \frac{p!}{(p-r)!} = p(p-1) \dots (p-r+1).$$

Now let n be an integer such that $s-1$ divides $n-1$ and

$$n \geq 3 + \frac{(s-1)(t+2)^{(4)}}{8}$$

Then $n-1 = (s-1)q$ for some positive integer q . Let there be given an edge coloring of K_n from any set of colors and suppose that no monochromatic $G_1 = K_{1,s}$ occurs. We show that there is a rainbow $G_2 = K_{t+1}$. Observe that the total number of different

copies of K_{t+1} in K_n is $\binom{n}{t+1}$ implying the existence of at least one rainbow K_{t+1} .

First consider the number of copies of K_{t+1} containing adjacent edges uv and uw that are colored the same. There are n possible choice for the vertex u . suppose that there are a_i edges incident with u that are colored i for $1 \leq i \leq k$. Then

$$\sum_{i=1}^k a_i = n-1,$$

Where, by assumption, $1 \leq a_i \leq s-1$ for each i . For each color i ($1 \leq i \leq k$), the number

of different choices for v and w where uv and uw are colored i is $\binom{a_i}{2}$. Hence the

number of different choices for u and w where uv and uw are colored the same is

$$\sum_{i=1}^k \binom{a_i}{2}$$

since the maximum value of this sum occurs when each a_i is as large as possible, the largest value of this sum is when each a_i is $s - 1$ and when $k = q$, that is, there are at most

$$\sum_{i=1}^q \binom{s-1}{2} = q \binom{s-1}{2}$$

Choices for v and w such that uv and uw are colored the same. Since there are $\binom{n-3}{t-2}$ choices for the remaining $t - 2$ vertices of K_{t+1} , it follows that there are at most

$$nq \binom{s-1}{2} \binom{n-3}{t-2}$$

copies of K_{t+1} containing two adjacent edges that are colored the same.

We now consider copies of K_{t+1} in which there two nonadjacent edges, say $e = xy$ and $f = wz$, colored the same. There are $\binom{n}{2}$ choices for e and $n - 2$ choices for one

vertex, say w , that is incident with f . The vertex w is incident with at most $s - 1$ edges having the same color as e and not adjacent to e . Since there are four ways of counting such a pair of edges in this way (namely e and either w or z , or f and either x or y), there are at most

$$\frac{\binom{n}{2}(n-2)(s-1)}{4} = \frac{n(n-1)(n-2)(s-1)}{8}$$

Ways to choose nonadjacent edges of the same color and $\binom{n-4}{t-3}$ ways to choose the remaining $t - 3$ vertices of K_{t+1} . Hence there are at most.

$$\frac{n(n-1)(n-2)(s-1)}{8} \binom{n-4}{t-3}$$

Copies of K_{t+1} containing two nonadjacent edges that are colored the same. Therefore, the number of non-rainbow copies of K_{t+1} is at most

$$nq \binom{s-1}{2} \binom{n-3}{t-2} + \frac{n(n-1)(n-2)(s-1)}{8} \binom{n-4}{t-3}$$

$$\begin{aligned}
&= \\
& n \binom{n-1}{s-1} \frac{(s-1)(s-2)}{2} \binom{n-2}{n-2} \binom{n-3}{t-2} \\
& + \frac{n(n-1)(n-2)(s-1)}{8} \binom{n-3}{n-3} \binom{n-4}{t-3} \\
&= \binom{n}{t+1} \left[\frac{(s-2)(t+1)^{(3)}}{2(n-2)} + \frac{(s-1)(t+1)^{(4)}}{8(n-3)} \right] \\
&< \binom{n}{t+1} \left[\frac{(s-1)(t+1)^{(3)}}{2(n-3)} + \frac{(s-1)(t+1)^{(4)}}{8(n-3)} \right] \\
&= \binom{n}{t+1} \left[\frac{(s-1)(t+1)^{(3)}(t+2)}{8(n-3)} \right] \\
&= \binom{n}{t+1} \left[\frac{(s-1)(t+2)^{(4)}}{8(n-3)} \right] \leq \binom{n}{t+1}
\end{aligned}$$

Where the final inequality follows from known theorem, the rainbow Ramsey number is defined if and only if F is a forest hence there is a rainbow K_{t+1} in K_n .

Acknowledgment

We would like to express our thanks to referees for valuable comments that improved the paper.

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