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4. THEOREMS ON STRUCTURE OF MINIMUM COUNTER EXAMPLE TO THE FIORINI-WILSON-FISK CONJECTURE

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Abstract:- In this Paper we will Prove some theorems on in structure of counter minimum counter example to the Fiorini-Wilson-Fisk Conjecture

Key Words:- Minimum Counter example, Unique colored graphs[1,2,3,4,5,6,7,8,9], Planar graphs, separating 4 – circuits, Separating 5- circuits.

Subject Classification: 05C15, 05C16 , 05C30

1. Introduction:-

Definitions and Notation

We start with some definitions to help us understand the discoveries that have been made about the structure of counter examples to the Fiorini- Wilson-Fisk Conjecture. A minimum counter example is graph G such that

(1) G is planar
(2) G is not a vertex Fiorini-Wilson-Fisk graph,
(3) G has at most one vertex-4-coloring, up to equivalence of colorings, and
(4) Subject to conditions (1), (2) and (3), $|v(G)|$ is minimum.

A set of edges $S \subset E$ of a graph an edge cut for a graph G if the graph G- S has at least two components. A graph is is K-edge-connected if every edge cut $S \subset E$ of G satisfies $/S/ \geq k$ An edge cut S of a graph G is cyclic if at least two components of G – S have circuits. A graph is cyclically k-edge-connected if every if every cyclic edge cut F satisfies $/S/ \geq k$ A cyclic edge-cut is trivial if one of the components of G – S is precisely a circuit containing / S /edges. Recall that a graph is k- connected if after the removal of any set having at most k-1 vertices, the graph is still connected. A graph G is internally 6- connected if it is 5- connected, and if for any set A of size 5 having the property that G –A is disconnected, it must be the case that one of the components of G –A consists of a single vertex.

Given a circuit C in a drawing (U,V) of a planar graph G, the Jordan Curve Theorem insures that C (treated as a curve in the topological sense) , partitions the sphere $\sum$ into two connected (topologically) regions which are homeomorphic to a disk. A circuit C in a drawing of a graph G is said to be a separating circuit of the drawing of g if when C, when considered as simple topological closed curve, has the property that the two arc-wise connected components of $\sum$ - C both contain vertices of V(G). When the drawing of G is clear from the context, we will also say that C is a separating circuit if G.

Work on the Fiorini-wilson-Fisk conjecture has produced a number of results about what a counter example to the conjecture must look like. It is not too difficult to prove that any counter example to the vertex version of the Fiorini-Wilson-Fisk Conjecture must be a triangulation of the plane, and that a minimum counter example must not have any separating triangles. Hind showed in his 1988 Ph.D. thesis that a minimum counter example to the edge version of the Fiorini-Wilson-Fisk Conjecture must have girth 5, which implies that vertices in a vertex counter example must have degree at least 5. Gold wasser and Zhang in strengthened this to prove that a minimum counter example to the edge version of the Fiorini-Wilson conjecture must be cyclically 5-edge-connected. They then showed in that every counter example to the edge version of the Fiorini-Wilson-Fisk conjecture must be cyuclically 5-edge-connected. They showed, moreover, that in a minimum counter example, every cyclic edge cut S , with/ S / =5 must be trivial. By considering the dual graph, this can be seen to imply that a minimum counter example as we have defined it in section 1 is Internally 6- connected. Boehme, stiebitxz and Voigt independently proved that a minimum counter example to the vertex version of the Fiorini-Wilson-Fisk Conjecture is 5- Connected.

We now give a proof of this result of GoldWasser and Zhang that says that a minimum counter example must be internally six- connected. First, we prove some theorems which show that the graph must be 5 – connected.

Theorem 1.1

Any counter example G to the Fiorini- wilson-Fisk Conjecture must have a drawing that is a triangulation. Moreover, if G is a minimum counter example, then no drawing of G which is a triangulation has a separating triangle.

Proof:

First let G be any counter example to the Fiorini-Wilson-Fisk Conjecture. The fact that some drawing of G is a triangulation of the plane follows from known Corollary [If G is a uniquely vertex 4-Colourable simple planar graph, then any drawing of the graph G is triangulation.Moreover,for $i \neq j$ and $i,j \in \{1,2,3,4\}$ each sub graph $G_{i,j}$ is a tree.]

Now let G be a minimum counter example, as in the definition found in section 1, Consider a drawing (U,V) of G which is a triangulation and which has a separating triangle C. Let $G^{'}$ be the drawing which consists of the sub drawing of (U,V) induced by all of the vertices that are in one of the arc-wise connected components of $\sum - C$ along with the all of the vertices of C. We see that $G^{'}$ uniquely 4- vertex colorable or else G would have two distinct vertex-4-colorings. Since G is assumed to be a minimum counter example then $G^{'}$ is a vertex Fiorini-Wilson-Fisk graph, and so theorem 2.8.4 implies that $G^{'}$ has at least two non-adjacent degree three vertices. Thus, one of these two degree 3 vertices must not be a vertex in

C, and therefore must also be a degree three vertex in the original graph G, which is a contradiction. This completes the proof of the theorem.

2. Excluding Separating 4- Circuits

If c is a proper vertex-4-coloring of a graph G which uses colors {1,2,3,4}, then we will denote by G(*i*,*j*,*c*) the sub graph of G induced by vertices colored *i* or j.If the coloring c is under stood from the context. Then we abbreviate G(*i*,*j*,*c*) by G(*i*.*j*). Throughout this section and section 3.
we will be assuming that G is a minimum counter example, and that C is a separating circuit in some drawing of (*U*,*V*) of G. We will denote by G_1 be the sub drawing of (*U*,*V*) included by all of the vertices in one of the arc-wise connected components of $\sum - C$ along with V(C),and we will denote by G_0 the sub drawing of (U,V) induced by $\{x_1, x_2, x_3, x_4\}$ and all of the vertices in other arc-wise connected component $\sum - C$ .We note that G_1 and G_0 each have a unique face bounded by C that is not a triangle, and so we can consider each of G_1 and G_0 to be a near triangulation by declaring this unique face in each to be the infinite face. We note that since G is a minimum counter example, then G_1,G_0 and graphs which are formed from G_1 or G_0 by either identifying vertices or adding only edges and which are also planar will be either vertex Fiorini-Wilson-Fisk graphs or will have at least two vertex-4-colorings. In either event, they will have at least one vertex-4-coloring.

Theorem2.1
Let G be a minimum counter example. No drawing of G has a seperating four circuit.
Proof: Suppose by way of contradiction that G has a drawing (*U*,*V*)with a separating four-circuit C having vertices x_1, x_2, x_3, x_4 with x_i adjacent to x_{i+1} for i=1,2,3 and x_4 adjacent to x_1. Note that $x_1$is not adjacent to x_3 nor is x_2 adjacent to x_4, because known theorem 1.1 guarantees the absence of separating triangles.

Let $C_I(C_0)$ denote the set of restrictions of four-colorings $G_I(G_0)$ to the four circuit { x_1, x_2, x_3, x_4 }. We will represent 4 – colorings of C by four element strings $a_1a_2a_3a_4$, where a_i is the color of x_i for $1 \le i \le 4.$ Two colorings c and c^1 of V(C) are deemed equivalent if there is a permutation $\prod$ of {1,2,3,4} such that $c(x_i)= \prod(c^1(x_i))$ for every $1 \le i \le 4.$ Let $C_1(C_0)$ denote those colorings in {1234,1213,1232,1212} that are equivalent to the restriction to $\{x_1, x_2, x_3, x_4\}$ of a four-colorings of $G_I(G_0)$.
We denote by C_{1234} those colorings of C which are equivalent to the colorings which assigns $C(x_1)=1$,$c(x_2)=2$,$c(x_3)=3$,and $c(x_4)=4$, and make analogous definitions for C_{1213},C_{1232}, and C_{1212}. Claim:

(i) $\{1212,1213\} \cap C_1 \ne \phi$

(ii) $\{1212,1232\} \cap C_1 \ne \phi$

(iii) $\{1232,1234\} \cap C_1 \ne \phi$

(iv) $\{1232,1234\} \cap C_1 \ne \phi$

<u>Proof of Claim.</u> To prove (i), form a graph from G_I by identifying x_1 and x_3. This graph is planar and is loop-less, because x_1 is not adjacent to x_3 in G. Since G is a

minimum counter example, this graph is either a vertex Fiorini-Wilson-Fisk graph or it has two vertex-4-colorings. Either way, it has at least one vertex-4-coloring. This 4-Coloring naturally corresponds to a 4- coloring of G_1 in which x_1 and x_3 receive the same color. The colorings which are equivalent to colorings in {1234, 1213, 1232, 1212} and that give x_1 and x_3 the same color are equivalent to the colorings 1212 and 1213. This shows that (i) holds. By similar reasoning applied to x_2 and x_4 we see that (ii) holds. To prove (iii), add the edge $\{x_1,x_3\}$ in the infinite face of G_1 and use the similar reasoning as above to produce a vertex -4-coloring of G_1 in which x_1 and x_3 receive different colors. Since all colorings for which x_1 and x_3 are colored differently are equivalent to 1232 and 1234, then (iii) holds. By similar reasoning applied to x_2 and x_4, (iv) holds. This completes the proof of the claim. The same proof could be applied to C_0 replaced by C_1.

By (i) and (iii) of the above claim. $|c_1| \geq 2$ and $|c_0| \geq 2$ If $|c_1| = 2$ then (i),(ii),(iii) and (iv) imply that c_1={1213,1232}. Similarly, if $|c_0|$ =2 then C_0={1234,1212} or C_0={1213,1232}.

We now show that we may assume $|c_1 \cap c_0| \geq 1.$ First note that unless $|c_1| = 2$ and $|c_0| = 2$ we would have $|c_1 \cap c_0| \geq 1.$ immediately. Thus we may assume that C_1={1234,1212} or C_1={1213,1232} and C_0 ={1213,1232} We may therefore assume without loss of generality that C_1={1234,1232}.

We may therefore assume without loss of generality that C_I={1234,1212} and C_0={1213,1232} or we would be done. Let c be a vertex-4-coloring of G_1 with the property that $c(x_1)=1,c(x_2)=2,c(x_3)=1$ and $c(x_4)=3$. If the vertices x_1 and x_3 are in the same component of the sub graph of $G_1(1,4,c)$, then the vertices x_2 and x_4 cannot be in the same component of the sub graph of $G_1(2,3,c)$ because G_1 is a drawing. Therefore, it is possible to interchange the colors 2 and 3 in the component of $G_1(2,3,c)$ which contains the vertex x_4. This will result in a coloring of G_1 which has restriction to C denoted by 1212 which is impossible because C_I={1213,1232}. It follows from this contradiction that the vertices x_1 and x_3 are in different components of G(1,4,c). Therefore, we may interchange the colors 1 and 4 in that component of G(1,4,c). which contains the vertex x_3 to get a coloring of G_1 whose restriction to C is given by 1243 and which can be made equivalent to the coloring 1234. This also contradicts C_I={1213,1232}. Thus, we may assume that $|c_1 \cap c_0| \geq 1.$

Now we complete the proof of the Theorem. Since $|c_1 \cap c_0| \geq 1.$, if $|c_1|$ =2 and $|c_0| = 2$ the work above shows that we must have $|c_1 \cap c_0| = 2$ since $|c_1| = 2$ implies that C_I={1234,1212} or C_I={1213,1232} with a similar statement for C_0. Also if $|c_I| \geq 3$ and $|c_0| \geq 3$. Then $|c_1 \cap c_0| \geq 2$ and we would be done. So we may assume by the symmetry of C_I and C_0 that $|c_I| = 2$ and $|c_0| = 3$. We will first show that $c_1 \neq \{1234{,}1212\}$. So assume that C_I={1234,1212}. Consider a four-coloring c of G_1 that has restriction to $\{x_1, x_2, x_3, x_4\}$ equivalent to 1212. Without loss of generality, we may assume that $c(x_1) = c(x_3) = 1$ and $c(x_2) = c(x_4) = 2$.

If there is a path in $G_I(1,2)$ joining x_1 and x_3, then the planarity of G_1 implies that there is no x_2-x_4 paths in G_1 whose vertices are colored 2 or 4. But then by interchanging colors in the component of the sub graph of $G_I(2,4)$ containing the vertex x_4, and permuting the colors 3 and 4, we see that $1213 \in c_I$ which is a contradiction. Thus, there is no x_1-x_3 path in G_I whose vertices are colored 1 and 3. Therefore, by interchanging the colors 1 and 3 in the component of $G_1(1,3,c)$ containing x_3, we see that $1232 \in c_I$ which again is a contradiction . This shows that $c_I \neq \{1234, 1212\}$

Assume then that $C_I=\{1213,1232\}$. Consider a 4- coloring of G_1 that has restriction to $\{x_1, x_2, x_3, x_4\}$ equivalent to 1213. By using reasoning similar to the previous paragraph, we can dispose of this case. This completes the proof of the Theorem.

3. Separating 5 – Circuits

Because of Theorem 1.1 and Theorem 2.1, it suffices to show that if C is a separating 5-circuit in a minimum counter example, then either the interior or the exterior of C has exactly one vertex. Let the vertices of such a separating circuit C of length 5 be x_1, x_2, x_3, x_4, x_5. Note that by Lemma 1.1 and Lemma 2.2 C is chord-less. We will introduce the concept of a minority vertex whose definition depends on whether C receives 3 or 4 colors in a proper 4- coloring. If c is a proper 4- coloring of C, in which C receives exactly 3 different colors, (which we assume are 1,2 and 3), then one of the colors {1,2,3} appears exactly once and the other two appear exactly twice. The vertex which receives the color which appears exactly once is called the minority vertex. If C receives four colors, say {1,2,3,4}, then exactly one of the colors {1,2,3,4} appears twice. The unique vertex in V(C) which is adjacent to the two vertices receiving this color will be the minority vertex in this case. By this definition, every 4-coloring of C is uniquely specified by stating which vertex in C is the minority vertex and whether C receives 3 or 4 colors. Thus we can denote a class of equivalent colorings of C by an ordered pair (i, α) where $i \in \{1,2,3,4,5\}$ and $\alpha \in \{3,4\}$ if x_i is the minority vertex in a 4-coloring of C belonging to this class which uses α colors. We will denote by C_I (respectively C_0) the set of ordered pairs of the form above which represent coloring classes of restrictions to C of 4- colorings of the graph G_1 (respectively G_0).

Lemma 3.1 Let $i \in \{1, \ldots, 5\}$

a) $If\,(i,3) \in C_I$,then $\{(i-1,3),(i-2,4)\} \cap C_I \neq \phi$ and $\{(i+1,3),(i+2,4)\} \cap C_I \neq \phi$

b) $If\,(i,4) \in C_I$, then $\{(i+3,3),(i+2,4)\} \cap C_I \neq \phi$, and $\{(i+2,3),(i+3,4)\} \cap C_I \neq \phi$

Here the first components of all ordered pairs should be interpreted modulo 5. The same statements hold for C_0 as well.

Proof: Let c be a coloring of G_I that has restriction to x_1,x_2,x_3,x_4,x_5 which is represented by (*i*,3). Without loss of generality we may assume (after possibly relabeling) that *i*=1 and x_1=1 and that $c(x_1)=1$, $c(x_2) = c(x_4)=2$ and $c(x_3)=c(x_5)=3$. If x_2 and x_4 are not in the same component of $G_1(2,4,c)$, then the colors 2 and 4 can be exchanged in the component containing x_4 to get a coloring of G_1 whose restriction to

x_1,x_2,x_3,x_4,x_5 is represented by (4,4). If x_2 and x_4 are in the same component of $G_1(2,4,c)$, then x_1 and x_3 are not in the same component of $G_1(1,3,c)$, because of they were, the path in this component would intersect the path joining x_2 and x_4 in $G_1(2,4)$. Therefore the colors in the component of $G_1(1,3,c)$ containing x_3 can be exchanged to get a coloring of G_1 whose restriction to x_1,x_2,x_3,x_4,x_5 is represented by (5,3). This proves the assertion that if $(i,3) \in C_I$ then $\{(i-1,3),(i-2,4)\} \bigcap C_I \neq \phi$.

Under the same assumption about c, if x_1 and x_4 are not in the same component of $G_I(1,2,c)$, then it can be shown that $(2,3) \in C_i$. If x_1 and x_4 are in the same component of G(1,2,c), then x_3 and x_5 must be in different components of $G_I(3,4,c)$, and by an appropriate interchange of the colors 3 and 4, it can be shown that $(3,4) \in C_i$. This establishes that $\{(i+1,3),(i+2,4)\} \bigcap C_I \neq \phi$.

The Proof of b) follows in the same way. Also, the same arguments can be applied to G_0 to reach the same conclusions about G_0. Therefore, the proof is complete.

The following notation will also be helpful: for *i*=1,2,3,4,5. Let $T_i = \{(i,4),(j,3),(k,3)\}$ and Q_i={(I,3), (j,4) , (k,4)}, where $j,k \in \{1,2,3,4,5\}$ and j $\equiv$i+2(mod 5) and k $\equiv$ i+3(mod 5). Arithmetic will be modulo 5, where the context demands.

This completes the proof of the theorem.

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References

[1]G.Chartrand, D. Geller, On uniquely colorable planar graphs, J. Combin.Theory 6 (1969) 271-278.

[2]A.G. Thomason, Hamiltonian cycles and uniquely edge colourable graphs, Ann. Discrete Math. 3 (1978) 259-268.

[3]Akbari, S. (2003). Two conjectures on uniquely totally colorable graphs. Discrete Math. *266(1-3), 41–45.*

[4]Akbari, S.; Behzad, M.; Haijiabolhassen, H.; Mahmoodian (1997). Uniquely total colorable graphs. *Graphs Combin.* **13**, 305–314.

[5]Bollobás, Béla (1978). *Extremal graph theory*, Vol. 11, LMS Monographs. London; New York; San Francisco: Academic Press.

[6]Hillar, C.; Windfeldt, T., An algebraic characterization of uniquely vertex colorable graphs, Journal of Combinatorial Theory Series B, 98 (2008), 400–414.

[7]Mahmoodian, E. S.; Shokrollahi, M. A. (1995). *Open problems at the combinatorics workshop of AIMC25 (Tehran, 1994)*, in Combinatorics Advances. In Colbourn, C. J.; Mahmoodian, E. S. (Eds.), Mathematics and its applications, 321–324. Dordrecht; Boston; London: Kluwer Academic Publishers.

[8]Truszczyński, M. (1981). *Some results on uniquely colourable graphs*. Soloquia Math. Soc. János Bolyai, **37**, 733–746.

[9]Xu, Shaoji (1990). The size of uniquely colorable graphs. *J. Combin. Theory (B)* **50**, 319–320.