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# 1.ON PLANAR COLOURED GRAPHS

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**Abstract**: In this paper we will work two proofs of the 4-Colour Theorem are presented which do not require the use of Computer Programs.

**Keywords**: Planar graph, near-triangulation, vertex coloring, chromatic number, 4-Colourable.

**Mathematics Subject Classification**: 05C15 , 05C30 , 05C75

**1.Introduction**

The Four Colour theorem states that every loop less planar graph admits a vertex four coloring. This was first proved by Appel and Haken [1] in 1976 with help of extensive computer calculations. An improved and independent version of this type of proof was given by Robertson et al[2]in 1996.The first proof of the corresponding list-colouring theorem for planar graphs –every planar graph is 5-choosable was provided by Thomassen[3] in 1994.This theorem is best possible-Voigt[4] found planar graphs which are not 4-choosable and S.Satyanarayana[5,6,7] Complete works on Four Colour Theorem(Research Note Book).

Proofs involving the extensive use of computer programs still unacceptable to some. Although the authors do not share this attitude, it is proposed that proofs which can be checked "by hand" should be preferred. Here two proofs of the 4-colour theorem are presented which fulfill this requirement. The proofs do not yield a colouring algorithm.

**2. Theorems for the colouring of planar graphs**

**Theorem 1: The chromatic number of a planar graph is not greater than 4**

We will work in the vertex-colouring context with the usual assumptions, i.e. coloured map in the plane or on a sphere is represented by its dual graph with coloured vertices. When two vertices are connected by an edge(i.e. the countries on the map have a common line-shaped border).a (proper) coloring requires that their colours be different. For convenience, let integers denote colours from colour lists with 4 coloures, which are the same for every vertex. sub list e.g. with 3 out of these 4 coloures are sometimes used for selected vertices.

Without loss of generality the proofs can be restricted to(planar)near-triangulations. A planar graph G is called a near-triangulation if it is connected, without loops, and every interior region is (bounded by) a triangle. A region is a triangle if it is incident with exactly 3 edges.

A full triangulation is the special case of a near-triangulation, in which also the infinite external region is bounded by a triangle(3-cycle).it follows from Euler's polyhedral formula that a planar graph with $n \geq 3$vertices has at most 3n-6 edges, and the triangulation is the

triangulation G by removing edges and disconnected vertices,i.e.H $\subseteq$ G.The removal of edges reduces the number of restrictions for coloruring,i.e.the chromatic number of H is not greater than the corresponding number of G.
Therefore, instead of theorem 1.we prove the following
***Theorem 2: Let G be a planar near-triangulation. A colouring of two adjacent vertices on outer cycle C $\subseteq$ G with colours 1 and 2 can be extended to a 3-colouring of C and a 4-colouring of its interior and hence of G.***

## 3. Proofs

In both proofs we perform induction with respect to the number n=/G/ of vertices. The cases n=3,4 are trival.Let n $\geq$ 4 and the theorem be true for n vertices. After the induction step G is a near-triangulation with n+ 1 vertex and at least 3 vertices on the outer cycle C.
Proof
If C has a chord S=$v_1v_2$, we colour its end- vertices with colours 1 and 2, respectively. The union of S with each of the two components of G –S yields the two induced sub graphs $G_1$ and $G_2$ with $G_1 \cap G_2 = S$ . Their outer cycles are $C_1=v_1v_2\ldots v_1 \subseteq G_1$ in clock wise and $C_2 = v_1v_2\ldots v_1 \subseteq G_2$ in counter clock wise order. Each with at least 3 vertices. Then we apply the induction hypothesis to both cycles and their interiors to obtain a 4- colouring of G.

Now We may assume that C has no chord. There is always a vertex $\overline{W}$ with K $\leq$ 5 neighbors, as the average vertex degree of a planar graph with n$\geq$3 vertices is d(G) $\leq$6-(12/n)<6, which follows from Euler’s polyhedral formula. Note that this result can be used to limit the valence of $\overline{W}$ to K$\leq$5, but – like in Thomason’s proof of the list – colouring theorem [3] – it turns out that its use is not necessary.

Let $\overline{W} \in$ G be a K-valent vertex and K$\geq$3 (e.g. K=5). Then the near- triangulation $G' = G - \overline{W}$ is of order n.

Let $\overline{W}$ be found on the outer cycle C=$V_1V_2\ldots V_p \overline{W} V_1$, p$\geq$2, with the vertices arrayed clock wise. We colour $V_1$ and $V_2$ with colours 1 and 2. Respectively. As C is chord less, the K- valent vertex $\overline{W}$ cannot be an end-vertex of a chord, and besides its neighbors $V_p$ and $V_1$ on C, it must have K-2 neighbors $V_{p+1},\ldots\ldots V_{p+k-2}$ not on C. As all interior regions are triangles, a path P = $V_pV_{p+1}\ldots\ldots V_{p+k-2}$ $v_1$ exists, and $C' = P \cup \left(C - \overline{w}\right)$ is the outer cycle of $G' = G - \overline{W}$ . Now we extend the coloured of $V_1$ and $V_2$ to a 3- colouring of $C'$ and to a 4- colouring of $G'$. Then $\overline{W}$ can be coloured with a colour not used for his k neighbours on C’. this concludes Proof 1.

**Proof 2**.

Let $\overline{W} \in G$ be a K- valent vertex, K $\geq$3, and $\overline{W} \notin C.$ We denote its adjacent vertices clock wise by $W_1,\ldots,W_k$. These vertices can be drawn on a unit circle (geometric circle with radius r=1) by stretching the drawing of the graph adequately. Now choose the special case of a (full) triangulation, i.e. P=3 vertices on C, by adding edges if necessary. Then inversion at the unit circle is applied: define the bijection

$$f : R^2 \setminus \{(0,0)\} \to R^2 \setminus \{(0,0)\} \quad \text{by} \quad f(r,\varphi) = \left(\frac{1}{r},\varphi\right).$$

Here r is the distance from the origin and $\varphi$ the angle in a polar coordinate system with origin (x,y)=(0,0) (this point can always be avoided in a drawing of the graph.) this bijection is continuous and maps the interior of the unit circle to its exterior and vice versa; points on the circle are fixed points. Note that any planar graph can be represented as a subset of the plane $R^2$: the vertices correspond to distinct point, the edges to simple Jordan curves ( which by re-drawing the graph can be chosen as straight line segments) between the points of their end – vertices. Hence *f* defines a correspondence between the two vertex sets *V* (g) and *f(V(G))* which preserves adjacency.

Now $G^n = f\left(G - \overline{w}\right)$ has n vertices and is a near – triangulation with an outer k- cycle $C^n$ = $w_1 w_{2.....} w_k\ w_1$. We may therefore assume that vertices $w_1$ and $w_2$ are coloured with colours 1 and 2, respectively, and that this colouring can be extended to a 3 – colouring of $C_n$ and a 4 – colouring of $G_n$.

The vertex $f(\overline{w})$ lies in the exterior region of $G^n$, with

$w_1 = f(w_1),....,w_k = f(w_k) \in C$ as its neighbours. Hence it can be coloured with a coloured with a colour not used on $C^{"}$. After colouring, as $f^{1} = f$, inversion at the unit circle is applied a second time in order to return to the original near – triangulation G. this concludes proof 2.

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